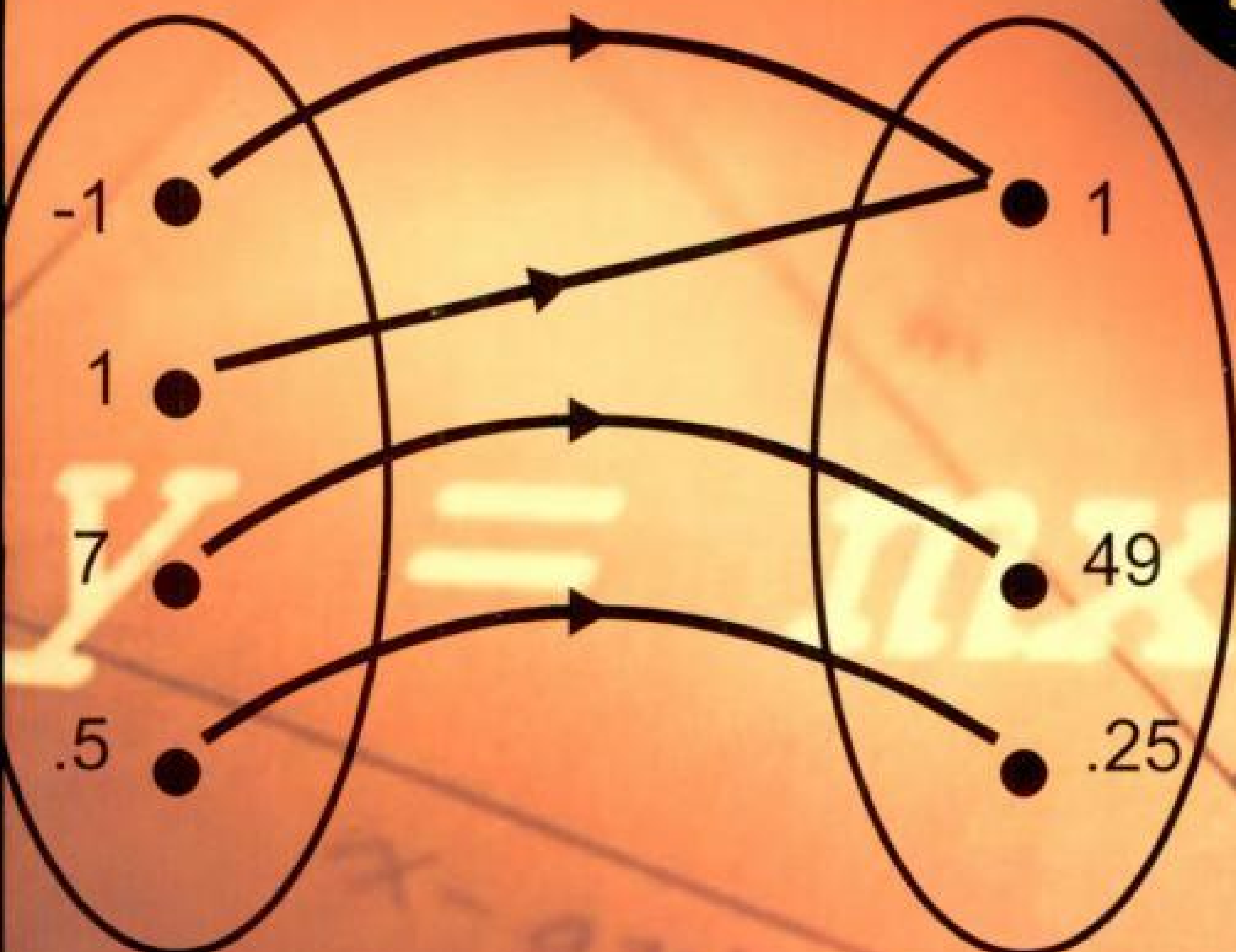
Eastern Economy Edition

CBSE Mathematics

(Thoroughly revised as per new syllabus)

For Class XI



Dinesh Khattar
Anita Khattar

PHI

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CBSE MATHEMATICS FOR CLASS XI
Dinesh Khattar and Anita Khattar

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PREFACE

This book has been written as a text for class XI of senior secondary schools (under the 10 + 2 pattern of education). The book is thoroughly revised and updated as per the new CBSE course structure and NCERT guidelines. It has been our experience that an average student needs the grasping of theory in a manner easily comprehensible to him or her. A concerted effort, therefore, has been made in this book to put concepts across in a lucid and unambiguous manner. It is particularly meant for beginners so as to enable them to grasp the fundamental concepts better.

The book is noteworthy in the following aspects:

- The subject matter is presented in a very systematic and logical manner. Every effort has been made to make the contents as simple and lucid as possible. Emphasis has been laid on making the concepts clear.
- The book contains large number of solved examples to develop problem solving techniques. The authors have put a lot of effort in the selection of examples. Care has been taken not to omit even a minor step so that the students can understand the subject without the guidance of a teacher. Hence, even an average student will be able to grasp the subject matter easily.
- Invariably it is observed that the level of questions in most of the books available is not enough for a student to pick up all of a sudden when they start preparing for engineering and other competitive examinations after their school exams. Keeping this limitation in view the problems have been categorized into two types—'Level of Difficulty A' and 'Level of Difficulty B'. Problems in the 'Level of Difficulty A' cover the needs of the students preparing for CBSE exams, whereas problems in the 'Level of Difficulty B' guide the students through more difficult problems for engineering entrance examinations. Thus, throughout the book the varying requirements of the average as well as the bright students are kept in mind.
- Working hints to a large number of problems have been provided at the end of each exercise. However, the authors would urge the students to attempt the same on their

own, failing which they should refer to the hints provided. The number of questions in each exercise have been kept sufficiently large to provide rigorous practice.

- Problem Solving Trick(s) are included to enhance and sharpen the problem solving skills. These tricks will also help to conquer the examination without sacrificing the mathematical accuracy.
- Each chapter contains 'Learning Objectives' which any student would like to achieve after having gone through that chapter. This will help the students focus their study.
- Common student difficulties and errors are highlighted under the heading 'CAUTION'.
- Each chapter is followed by a 'Chapter Test', which includes problems related to all the topics, so that students have full understanding of these topics before going to the next chapter.

The answers to almost all the unsolved problems have been checked and every care has been taken to minimize printing and other errors. It is earnestly hoped that this book will help the students score 100% marks in CBSE and at the same time build a strong foundation for success in any competitive examination.

We wish to place on record our sincere thanks to our friends and colleagues for their help and suggestions for planning and preparing the manuscript of this book.

This work would not have been possible without the support and encouragement from Prentice-Hall of India. My sincere thanks and words of appreciation to all of them, who are directly or indirectly involved in the project.

Suggestions and comments to improve the book in content and style are always welcome and will be greatly appreciated and acknowledged.

Thank you for choosing our book.

May you find it stimulating and rewarding!

Authors

SYLLABUS

Units

I. Sets and Functions
II. Algebra
III. Coordinate Geometry
IV. Calculus
V. Mathematical Reasoning
VI. Statistics and Probability

Appendices

1. Infinite Series
2. Mathematical Modelling

UNIT I: SETS AND FUNCTIONS

1. Sets **(12 Periods)**

Sets and their representations. Empty set. Finite and Infinite sets. Equal sets. Subsets. Subsets of the set of real numbers especially intervals (with notations). Power set. Universal set. Venn diagrams. Union and Intersection of sets. Difference of sets. Complement of a set.

2. Relations and Functions **(14 Periods)**

Ordered pairs, Cartesian product of sets. Number of elements in the cartesian product of two finite sets. Cartesian product of the reals with itself (upto $R \times R \times R$).

Definition of relation, pictorial diagrams, domain, co-domain and range of a relation. Function as a special kind of relation from one set to another. Pictorial representation of a function, domain, co-domain and range of a function. Real valued function of the real variable, domain and range of these functions, constant, identity, polynomial, rational, modulus, signum and greatest integer functions with their graphs. Sum, difference, product and quotients of functions.

3. Trigonometric Functions **(18 Periods)**

Positive and negative angles. Measuring angles in radians and in degrees and conversion from one measure to another. Definition of trigonometric functions with the help of unit circle. Truth of the identity $\sin^2x + \cos^2x = 1$, for all x. Signs of trigonometric functions and sketch of their graphs. Expressing $\sin(x+y)$ and $\cos(x+y)$ in tems of $\sin x$, $\sin y$, $\cos x$ and $\cos y$. Deducing the identities like following:

$$\tan(x+y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}, \cot(x \pm y) = \frac{\cot x \cot y \mp 1}{\cot y \pm \cot x},$$

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}, \cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2},$$

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}, \cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}.$$

Identites related to $\sin 2x$, $\cos 2x$, $\tan 2x$, $\sin 3x$, $\cos 3x$ and $\tan 3x$. General solution of trigonometric equations of the type $\sin\theta = \sin\alpha$, $\cos\theta = \cos\alpha$ and $\tan\theta = \tan\alpha$. Proofs and simple applications of sine and coine formulae.

UNIT II: ALGEBRA

1. Principle of Mathematical Induction **(06 Periods)**

Processes of the proof by induction, motivating the application of the method by looking at natural numbers as the least inductive subset of real numbers. The principle of mathematical induction and simple applications.

2. Complex Numbers and Quadratic Equations **(10 Periods)**

Need for complex numbers especially $\sqrt{-1}$, to be motivated by inability to solve every quadratic equation. Brief description of algebraic properties of complex numbers. Argand plane and polar representation of complex numbers. Statement of Fundamental Theorem of Algebra, solution of quadratic equations in the complex number system.

3. Linear Inequalities **(10 Periods)**

Linear inequalities. Algebraic solutions of linear inequalities in one variable and their representation on the number line. Graphical solution of linear inequalities in two variables. Solution of system of linear inequalities in two variables–graphically.

4. Permutations and Combinations **(12 Periods)**

Fundamental principle of counting. Factorial n. Permutations and combinations, derivation of formulae and their connections, simple applications.

5. Binomial Theorem **(08 Periods)**

History, statement and proof of the binomial theorem for positive integral indices. Pascal's triangle, general and middle term in binomial expansion, simple applications.

6. Sequence and Series **(10 Periods)**

Sequence and Series. Arithmetic progression (A.P.), artithmetic mean (A.M.). Geometric progression (G.P.), general term of a G.P., sum of n terms of a G.P., geometric mean (G.M.), relation between A.M. and G.M. Sum to n terms of the special series: $\sum n, \sum n^2$ and $\sum n^3$.

UNIT III: COORDINATE GEOMETRY

1. Straight Lines **(09 Periods)**

Brief recall of 2D from earlier classes. Slope of a line and angle between two lines. Various forms of equations of a line: parallel to axes, point-slope form, slope-intercept form, two-point form, intercepts form and normal form. General equation of a line. Distance of a point from a line.

2. Conic Sections **(12 Periods)**

Sections of a cone: Circles, ellipse, parabola, hyperbola, a point, a straight line and pair of intersecting lines as a degenerated case of a conic section. Standard equations and simple properties of parabola, ellipse and hyperbola. Standard equation of a circle.

3. Introduction to Three-dimensional Geometry **(08 Periods)**

Coordinate axes and coordinate planes in three dimensions. Coordinates of a point. Distance between two points and section formula.

UNIT IV: CALCULUS

1. Limits and Derivatives **(18 Periods)**

Derivative introduced as rate of change both as that of distance function and geometrically, intuitive idea of limit. Definition of derivative, relate it to slope of tangent of the curve, derivative of sum, difference, product and quotient of functions. Derivatives of polynomial and trigonometric functions.

UNIT V: MATHEMATICAL REASONING **(08 Periods)**

Mathematically acceptable statements. Connecting words/phrases—consolidating the understanding of "if and only if (necessary and sufficient) condition", "implies", "and/or", "implied by", "and", "or", "there exists" and their use through variety of examples related to real life and Mathematics. Validating the statements involving the connecting words—difference between contradiction, converse and contrapositive.

UNIT VI: STATISTICS AND PROBABILITY

1. Statistics **(10 Periods)**

Measure of dispersion; mean deviation, variance and standard deviation of ungrouped/grouped data. Analysis of frequency distributions with equal means but different variances.

2. Probability **(15 Periods)**

Random experiments: Outcomes, sample spaces (set representation). Events: Occurrence of events, 'not', 'and' & 'or' events, exhaustive events, mutually exclusive events. Axiomatic (set theoretic) probability, connections with the theories of earlier classes. Probability of an event, probability of 'not', 'and' & 'or' events.

APPENDICES

1. Infinite Series

Binomial theorem for any index, infinite geometric series, exponential and logarithmic series.

2. Mathematical Modelling

Consolidating the understanding developed up to Class X. Focus on modelling problems related to real-life (like environment, travel, etc.) and connecting with other subjects of study where many constraints may really need to be ignored, formulating the model, looking for solutions, interpreting them in the problem situation and evaluating the model.

CHAPTER

1

SETS

Learning Objectives

After successful completion of this chapter, the reader should be able to understand and appreciate:

- Sets and their Representations
- The Empty Set
- Finite and Infinite Sets
- Subsets
- Equal Sets
- Power Set
- Universal Set
- Venn Diagrams
- Operations on Sets
- Complement of a Set
- Practical Problems on Union and Instersection of Two Sets

INTRODUCTION

The concept of set is fundamental in all branches of mathematics. Sets are the most basic tools of mathematics which are extensively used in developing the foundations of relations and functions, logic theory, sequences and series, geometry, probability theory, etc. In fact, these days most of the concepts and results in mathematics are expressed in the set theoretic language.

The modern theory of sets was developed by German mathematician George Cantor (1845–1918 A.D.). In this chapter, we shall study some basic definitions and operations involving sets. We shall also discuss the applications of sets.

SET

We observe that in nature varieties of objects occur in *groups*. These groups are given different names such as a *collection* of books, a *bunch* of keys, a *herd* of cattle, an *aggregate* of points, etc., depending on the characteristic of objects they represent. In literal sense all these words have the same meaning (i.e., a group or a collection). In mathematical language, we call this collection of objects, a *Set*. From the above examples it can be seen that each collection has a well defined property (characteristic) of its own.

Thus, a *set* is a *well defined collection of objects*. When we say well-defined, we mean that the objects follow a given rule or rules. With the help of this rule, we will be able to say whether any given object belongs to this set or not. For example, if we say that we have a collection of short students in a class, then this collection is not a set as "short students" is not well-defined. However, if we say that we have a collection of students whose height is less than 5 feet, then it represents a set.

It is not necessary that a set may consist of same type of objects. For example, a book, a cup and a plate lying on a table may also form a set, their common property being that they form a collection of objects *lying on the table.*

Illustration 1. Some other examples of sets are:

(i) The set of numbers 1, 3, 5, 7, 9, 14
(ii) The set of vowels in the alphabets of English
(iii) The set of rivers in India
(iv) The set of all planets
(v) The set of points on a circle
(vi) The set of mathematics books in your library
(vii) The set of even positive integers (i.e., 2, 4, 6, 8....)
(viii) The set of multiples of 4 (i.e., 4, 8, 12...)
(ix) The set of factors of 12 (i.e., 1, 2, 3, 4, 6, 12).
(x) The set of integers less than zero (i.e., – 1, – 2, – 3, ...)

Notations

Sets are usually denoted by capital letters A, B, C, etc. and their elements by small letters a, b, c, etc.

Let A be any set of objects and let 'a' be a member of A, then we write $a \in A$ and read it as 'a belongs to A' or 'a is an *element* of A' or 'a is a *member* of A'. If a is not an object of A, then we write $a \notin A$ and read it as 'a does not belong to A' or 'a is not an element of A' or 'a is not a member of A'.

REPRESENTATION OF SETS

There are two ways of expressing a set. These are:

1. Tabular Form or Roster Form.
2. Set Builder Form or Rule Method

Tabular Form or Roster Form

In this method we list all the members of the set separating them by means of commas and enclosing them in curly brackets { }.

Illustration 2. Let A be the set consisting of the numbers 1, 3, 4 and 5, then we write $A = \{1, 3, 4, 5\}$.

CAUTION

- The order of writing the elements of a set is immaterial. For example, {1, 3, 5}, {3, 1, 5}, {5, 3, 1} all denote the same set.
- An element of a set is not written more than once. Thus, the set {1, 5, 1, 3, 4, 1, 4, 5} must be written as {1, 3, 4, 5}.

Set-Builder Form or Rule Method

In this method, instead of listing all elements of a set, we write the set by some special property or properties satisfied by all its elements and write it as:

$$A = \{x : P(x)\} \quad \text{or} \quad A = \{x \mid x \text{ has the property } P(x)\}$$

and read it as 'A is the set of all elements x such that x has the property P'. The symbol ' : ' or ' | ' stands for 'such that'.

Illustration 3. Let A be the set consisting of the elements 2, 3, 4, 5, 6, 7, 8, 9, 10. Then the set A can be written as:

$$A = \{x : 2 \le x \le 10 \text{ and } x \in N\}$$

Example 1. *Represent the following sets in the roster form:*

(i) $A = \{x : x$ *is a natural number less than* $6\}$

(ii) $B = \{x : x$ *is an integer and* $-3 \le x < 7\}$

(iii) $C = \{x : x$ *is a two-digit number such that the sum of its digits is* $8\}$

(iv) $D = \{x : x$ *is a prime number which is a divisor of* $60\}$

(v) $E =$ *the set of all letters in the word* TRIGONOMETRY

(vi) $F =$ *the set of all letters in the word* BETTER

Solution: (i) $A = \{1, 2, 3, 4, 5\}$

(ii) $B = \{-3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$

(iii) $C = \{17, 26, 35, 44, 53, 62, 71, 80\}$

(iv) $D = \{2, 3, 5\}$

(v) $E = \{$T, R, I, G, O, N, M, E, Y$\}$

(vi) $F = \{$B, E, T, R$\}$

Example 2. *Represent the following sets in the set builder form:*

(i) $A = \{2, 3, 5, 7, 11, 13, 17, 19, ...\}$ (ii) $B = \{2, 3, 5\}$

(iii) $C = \{1, 2, 3, 4, 6, 12, 24, 48\}$ (iv) $D = \{$a, e, i, o, u$\}$

(v) $E = \{0, 3, 6, 9, 12, 15, 18, ...)$ (vi) $F = \{1, 5, 10, 15, ...)$

(vii) $G = \{14, 21, 28, 35, 42, ..., 98\}$

(viii) $H = \{-1, 1\}$

Solution: (i) $A = \{x : x$ is a prime number$\}$

(ii) $B = \{x : x$ is a prime divisor of $30\}$

(iii) $C = \{x : x$ is a divisor of $48\}$

(iv) $D = \{x : x$ is a vowel of the English alphabet$\}$

(v) $E = \{x : x = 3n$, n is a whole number$\}$

(vi) $F = \{x : x$ is a natural number multiple of 5 and $x = 1\}$

(vii) $G = \{x : x$ is a natural number multiple of 7 and $7 < x < 100\}$

(viii) $H = \{x : x$ is an odd integer and $|x| < 2\}$

Example 3. *Write the following sets in the set builder form:*

(i) $A = \{41, 43, 47\}$ (ii) $B = \{7, 8, 9, 10, 11\}$ (iii) $C = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\right\}$

(iv) $D = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right\}$ (v) $E = \left\{1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots\right\}$

(vi) $F = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}, \frac{9}{10}\right\}$

Solution: (i) $A = \{x : x$ is a prime number between 40 and $50\}$

(ii) $B = \{x : x$ is a natural number between 6 and $12\}$

(iii) $C = \{x : x = \frac{1}{n}$, where n is a natural number$\}$

(iv) $D = \{x : x = \frac{n}{n+1}, n \in N\}$

(v) $E = \{x : x \frac{1}{n^2}$, where n is a natural number$\}$

(vi) $F = \left\{x : x = \frac{n}{n+1}, n \in N, n \leq 9\right\}$

Example 4. *Let* $A = \{1, 2, 3, 4\}$, $B = \{5, 6, 7\}$ *and* $C = \{7, 8, 9, 10\}$. *Insert the correct symbol* $\in$ *or* $\notin$ *in each of the following blanks:*

(i) 2 ... A (ii) 5 ... C (iii) 6 ... B

(iv) 7 ... A (v) 5 ... B (vi) 10 ... C

Solution: (i) $2 \in A$ (ii) $5 \notin C$ (iii) $6 \in B$

(iv) $7 \notin A$ (v) $5 \in B$ (vi) $10 \in C$

Example 5. *Match each of the following sets in the column A described in roster form with the same set in the column B described in the set-builder form.*

(i) {F, L, O, W} (a) $\{x : x$ *is a positive integer and is a divisor of* $18\}$

(ii) $\{1, 4, 9, 16, 25, 36\}$ (b) $\{x : x$ *is an integer and* $x + 1 = 1\}$

(iii) $\{-3, 3\}$ (c) $\{x : x$ *is a letter of the word* FOLLOW$\}$

(iv) $\{11, 13, 17, 19\}$ (d) $\{x : x = n^2, n \in N$ *and* $x < 40\}$

(v) $\{0\}$ (e) $\{x : x$ *is an integer and* $x^2 - 9 = 0\}$

(vi) $\{1, 2, 3, 6, 9, 18\}$ (f) $\{x : x$ *is a prime number and* $11 \leq x \leq 19\}$

Solution: (i) $\leftrightarrow$ (c); (ii) $\leftrightarrow$ (d); (iii) $\leftrightarrow$ (e); (iv) $\leftrightarrow$ (f); (v) $\leftrightarrow$ (b); (vi) $\leftrightarrow$ (a)

EXERCISE 1.1

1. Which of the following are sets?

(i) The collection of all months of a year beginning with letter J;

(ii) The collection of ten most talented writers of India;

(iii) A team of eleven best cricket batsmen of the world;

(iv) The collection of all boys in your class;

(v) The collection of all natural numbers less than 100;

(vi) A collection of novels written by the writer Munshi Prem Chand;

(vii) The collection of all even integers;

(viii) The collection of different problems in this chapter;

(ix) A collection of most dangerous animals of the world.

2. Let $A = \{x \mid x \in N$ and $4 < x < 12\}$. Which of the following statements are true and which are false?

(i) $2 \in A$; (ii) $8 \notin A$; (iii) $16 \in A$; (iv) $7 \in A$.

3. Let $A = \{7, 8, 11, 13\}$, $B = \{2, 4, 9\}$ and $C = \{1, 3, 4, 8\}$. Insert the correct symbol $\in$ or $\notin$ in each of the following blanks:

(i) 7 ... A; (ii) 8 ... B; (iii) 9 ... C;

(iv) 4 ... B; (v) 9 ... A; (vi) 4 ... C.

4. Write the following sets in set-builder form.

(i) $A = \{3, 6, 9, 12\}$; (ii) $B = \{2, 4, 8, 16, 32\}$;

(iii) $C = \{5, 25, 125, 625\}$; (iv) $D = \{2, 4, 6, ...\}$;

(v) $E = \{1, 4, 9, ..., 100\}$.

5. Write the following sets in the tabular form:

(i) $A = \{x \mid x^2 = 16\}$;
(ii) $B = \{x \mid x$ is positive, x is negative$\}$;
(iii) $C = \{x \mid x$ is a letter in the word INDIA$\}$;
(iv) $D = \{x \mid x$ is a day of the week$\}$;
(v) $E = \{x \mid x$ is an even natural number$\}$.

6. List the members of the following sets:

(i) $S = \{x \mid x$ is a prime number less than $30\}$;
(ii) $S = \{x \mid x$ is a prime number less than 50 and the digit in its unit's place is $1\}$;
(iii) $S = \{x \mid x$ is the square of the natural number and $x < 50\}$.

7. Rewrite the following sets in roster form:

(i) $\{x \mid x$ is a letter in the word ASSASSINATION$\}$;
(ii) $\{x \mid x \in N$, x is odd and $x < 10\}$;
(iii) $\{x \mid \in N$, x is a multiple of 7 and $x < 60\}$;
(iv) $\{x \mid x \geq 0$ and $x^2 = 4\}$;
(v) $\{x \mid x \in Z, x^2 < 20\}$.

8. List the elements of the following sets:

(i) $A = \{x : x$ is an odd natural number$\}$;
(ii) $B = \{x : x$ is an integer, $-\frac{1}{2} < x < \frac{9}{2}\}$;
(iii) $C = \{x : x$ is an integer, $x^2 \leq 4\}$;
(iv) $D = \{x : x$ is a letter in the word LOYAL$\}$;
(v) $E = \{x : x$ is a month of the year not having 31 days$\}$;
(vi) $F = \{x : x$ is a consonant in the English alphabet which precedes k$\}$.

9. Match each of the sets on the left in the roster form with the same set on the right described in the set builder form:

(i) $\{1, 2, 3, 6\}$ (a) $\{x : x$ is a prime number and a divisor of $6\}$
(ii) $\{2, 3\}$ (b) $\{x : x$ is an odd natural number less than $10\}$
(iii) $\{$M, A, T, H, E, I, C, S$\}$ (c) $\{x : x$ is a natural number and divisor of $6\}$
(iv) $\{1, 3, 5, 7, 9\}$ (d) $\{x : x$ is a letter of the word MATHEMATICS$\}$

10. Use roster method to express each of the following sets:

(i) $A = \{x : x = n^3, n \in N$ and $x < 80\}$; (ii) $B = \{x : x \in I$ and $-5 < x < 2\}$;
(iii) $C = \{(x, y) : x, y \in N$ and $x = y\}$; (iv) $D = \{x : x \in$ W and x is even prime$\}$.

11. Write the following sets in set builder form:

(i) $A = \{2, 5, 7\}$; (ii) $B = \{2, 9, 28, 65, 126\}$; (iii) $C = \{-7, 7\}$;
(iv) $D = \{2, -3\}$; (v) $E = \{0\}$; (vi) $F = \left\{\frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}\right\}$;
(vii) $G = \left\{\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \frac{5}{26}, \frac{6}{37}, \frac{7}{50}\right\}$.

12. Express the following sets in roster form:

(i) $A = \{x : x$ is a factor of $12\}$;
(ii) $B = \{x : x$ is a root of $x^3 - 6x^2 + 11x - 6 = 0\}$;
(iii) $C = \{x : x$ is a month beginning with M$\}$;
(iv) $D = \{x : x$ is a letter of the word MISSISSIPPI$\}$;
(v) $E = \{x : x$ is a woman Prime Minister of India$\}$;
(vi) $F = \left\{x : x = \frac{n^2 - 1}{n^2 + 1}, n \in N \text{ and } n < 4\right\}$.

13. Express the following sets in set builder form:

(i) $A = \{0, 3, 8, 15, 24\}$; (ii) $B = \{2, 9, 28, 65\}$; (iii) $C = \left\{\frac{1}{5}, \frac{1}{3}, \frac{3}{7}, \frac{1}{2}\right\}$;

(iv) D = {January, June, July}; (v) $E = \{10, 21, 32, 43\}$; (vi) $F = \{12, 13, 14\}$.

14. Match each of the sets on the left expressed in roster form with the same set described in set builder form on the right:

(i) $\{2, 3, 5\}$ (a) $\{x : x$ is a factor of $14\}$

(ii) {C, E, I, M, T} (b) $\{x : x$ is a root of $x^2 - 5x + 6 = 0\}$

(iii) $\{2, 3\}$ (c) $\{x : x$ is the largest prime number less than $10\}$

(iv) $\{0, 15, 80\}$ (d) $\{x : x$ is a letter of the word COMMITTEE$\}$

(v) $\{1, 2, 7, 14\}$ (e) $\{x : x$ is a prime number less than $7\}$

(vi) $\{7\}$ (f) $\{x : x = n^4 - 1, n \in N, n < 4\}$.

15. Write the following sets in roster form:

(i) $A = \{a_n \mid n \in N, a_{n+1} = 3a_n$ and $a_1 = 2\}$;

(ii) $B = \{a_n \mid n \in N, a_{n+2} = a_{n+1} + a_n, a_1 = a_2 = 1\}$.

16. Describe the following sets by roster method:

(i) $\{x : x^3 - 1 = 0, x \in R\}$; (ii) $\{x : |x| < 3, x \in Z\}$;

(iii) $\{x : x^2 < 9, x \in Z\}$; (iv) $\{x : x^2 > 9, x \in Z\}$.

17. Let $A = \{1, 2, 3, 4, 5, 6\}$. Insert the appropriate symbol $\in$ or $\notin$ in the blank spaces.

(i) 5 ... A (ii) 8 ... A (iii) 0 ... A

(ii) 4 ... A (v) 2 ... A (vi) 10 ... A

Answers

1. (i); (iv); (v); (vi); (vii); and (viii) are sets.

2. (i) False; (ii) False; (iii) False; (iv) True.

3. (i) $\in$; (ii) $\notin$; (iii) $\notin$; (iv) $\in$; (v) $\notin$; (vi) $\in$.

4. (i) $A = \{x : x$ is a natural number multiple of 3 and $x < 15\}$;

(ii) $B = \{x : x = 2^n, n \in N$ and $n < 6\}$;

(iii) $C = \{x : x = 5^n, n \in N$ and $n \geq 4\}$;

(iv) $D = \{x : x$ is an even natural number$\}$;

(v) $E = \{x : x = n^2, n \in N$ and $N < 11\}$.

5. (i) $\{-4, 4\}$; (ii) ϕ; (iii) {I, N, D, A};

(iv) {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday};

(v) $E = \{2, 4, 6, 8, ...\}$.

6. (i) $S = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$;

(ii) $S = \{11, 31, 41\}$;

(iii) $S = \{1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2\} = \{1, 4, 9, 16, 25, 36, 49\}$.

7. (i) {A, S, I, N, T, O}; (ii) $\{1, 3, 5, 7, 9\}$;

(iii) $\{7, 14, 21, 28, 35, 42, 49, 56\}$; (iv) $\{2\}$;

(v) $\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$.

8. (i) $A = \{1, 3, 5, 7, ...\}$; (ii) $B = \{0, 1, 2, 3, 4\}$;

(iii) $C = \{-2, -1, 0, 1, 2\}$; (iv) D = {L, O, Y, A};

(v) E = {February, April, June, September, November};

(vi) $F = \{b, c, d, f, g, h, j\}$.

9. (i) $\leftrightarrow$ (c); (ii) $\leftrightarrow$ (a); (iii) $\leftrightarrow$ (d); (iv) $\leftrightarrow$ (b).

10. (i) $A = \{1, 8, 27, 64\}$; (ii) $B = \{-4, -3, -2, -1, 0, 1\}$;
(iii) $C = \{(1, 1), (2, 2), (3, 3), ...\}$; (iv) $D = \{2\}$.

11. (i) $A = \{x : x$ is a prime number less than $10\}$;
(ii) $B = \{x : x = n^3 + 1, n \in N$ and $n \leq 5\}$;
(iii) $C = \{x : x^2 - 49 = 0\}$;
(iv) $D = \{x : x^2 + x - 6 = 0\}$;
(v) $E = \{x : x$ is the smallest whole number$\}$;
(vi) $F = \{x : x = \frac{n}{n+2}, n \in N, n \leq 4\}$; (vii) $\left\{\frac{n}{n^2+1} ; n \in N, n \leq 7\right\}$.

12. (i) $\{1, 2, 3, 4, 6, 12\}$; (ii) $\{1, 2, 3\}$; (iii) {March, May};
(iv) {M, I, S, P}; (v) {Mrs. Indira Gandhi}; (vi) $\left\{0, \frac{3}{5}, \frac{4}{5}\right\}$.

13. (i) $A = \{x : x = n^2 - 1, n \in N$ and $n \leq 5\}$;
(ii) $B = \{x : x = n^3 + 1, n \in N$ and $n \leq 4\}$;
(iii) $C = \{x : x = \frac{n}{n+4}, n \in N$ and $n < 5\}$;
(iv) $D = \{x : x$ is a month of the year beginning with $J\}$;
(v) $E = \{x : x = 11n - 1, n \in N$ and $n \leq 4\}$;
(vi) $F = \{x : x \in N, 12 \leq x \leq 14\}$.

14. (i) $\leftrightarrow$ (e); (ii) $\leftrightarrow$ (d); (iii) $\leftrightarrow$ (b); (iv) $\leftrightarrow$ (f); (v) $\leftrightarrow$ (a); (vi) $\leftrightarrow$ (c).

15. (i) $\{2, 6, 18, 54, ...\}$; (ii) $\{1, 2, 3, 5, 8, ...\}$.

16. (i) $\{1\}$; (ii) $\{-2, -1, 0, 1, 2\}$; (iii) $\{-2, -1, 0, 1, 2\}$;
(iv) $\{-4, 4, -5, 5, -6, 6, ...\} = Z - \{-3, -2, -1, 0, 1, 2, 3\}$.

17. (i) $5 \in A$; (ii) $8 \notin A$; (iii) $0 \notin A$; (iv) $4 \in A$; (v) $2 \in A$; (vi) $10 \notin A$.

HINTS AND SOLUTIONS

1. (i) The members of this collection are: January, June and July. Clearly, this collection is well-defined and hence, it is a set.

(ii) Th term 'most talented' is a vague term. A writer may be most talented to one person and may not be so to the other. Thus, we cannot judge definitely which writers are there in this collection. Therefore, this collection is not well-defined and hence, it is not a set.

(iii) The term 'best' is a vague term. A batsman may be best to one person and may not be so to the other. Thus, we cannot judge definitely which batsmen are there in this collection, Therefore, this collection is not well-defined and hence, it is not a set.

(iv) We can definitely say that the members of this collection are your class-fellows. Therefore, this collection is well-defined and hence, it is a set.

(v) The members of the collection are 1, 2, 3, …, 97, 98, 99. Therefore, this collection is well-defined and hence, it is a set.

(vi) The members of the collection are the novels written by Munshi Prem Chand. Therefore, this collection is well-defined and hence, it is a set.

(vii) The members of this collection are 2, 4, 6, … . Clearly, this collection is well-defined and hence, it is a set.

(viii) Clearly, the members of the collection are the different problems of this chapter. So, it is well-defined and hence, it is a set.

(ix) The term 'most dangerous' is a vague term. An animal may be most dangerous to one person and may not be so to the other. Thus, it is not well-defined. So, it is not a set.

FINITE AND INFINITE SETS

Finite Set

A set having no element or a definite number of elements is called a *finite set.*

Thus, in a finite set, either there is nothing to be counted or the number of elements can be counted, one by one, with the counting process coming to an end.

Illustration 1. Each of the following sets is a finite set:

(i) A = the set of prime numbers less than 10 = {2, 3, 5, 7};
(ii) B = the set of vowels in English alphabet = {a, e, i, o, u,};
(iii) $C = \{x \mid x$ is divisor of $50\}$.

Cardinal Number of a Finite Set

The number of distinct elements in a finite set S is called the *cardinal number* of S and is denoted by $n(S)$.

Illustration 2. If $A = \{2, 4, 6, 8\}$, then $n(A) = 4$.

Infinite Set

A set having unlimited number of elements is called an *infinite set*. Thus, in an infinite set, if the elements are counted one by one, the counting process never comes to an end.

Illustration 3. Each of the following sets is an infinite set:

(i) The set of all natural numbers = {1, 2, 3, 4, ...};
(ii) The set of all prime numbers = {2, 3, 5, 7, ...};
(iii) The set of all points on a given line;
(iv) The set of all lines in a given plane;
(v) $\{x \mid x \in R \text{ and } 0 < x < 1\}$.

CAUTION

All infinite sets cannot be described in the roster form. For example, the set of real numbers cannot be described in this form, because the elements of this set do not follow any particular pattern.

EMPTY SET (OR NULL SET)

The set which contains no element is called the *empty set* or the *null set* or *void set.*

The symbol for the empty set or the null set is ϕ. Thus, $\phi = \{\}$, since there is no element in the empty set.

The empty set is a finite set.

Since any object x which is not equal to itself does not exist, therefore, the set $A = \{x : x \neq x\}$ is the empty set ϕ.

A set which is not empty, i.e., which has atleast one element is called a *non-empty* set or a *non-void set.*

Illustration 4.

(i) The set of natural numbers less than 1 is the empty set;
(ii) The set of odd numbers divisible by 2 is the null set;
(iii) $\{x \mid x \in Z \text{ and } x^2 = 2\} = \phi$, because there is no integer whose square is 2;
(iv) $\{x \mid x \in R \text{ and } x^2 = -1\} = \phi$, because the square of a real number is never negative;
(v) $\{x \mid x \in N, 4 < x < 5\}$ is the empty set;
(vi) $\{x \mid x \in Z, -1 < x < 0\}$ is the null set.

The empty set should not be confused with the set $\{0\}$. It is the set containing one element, namely 0.

SINGLETON

A set containing only one element is called a *singleton.*

Illustration 5.

(i) The set $\{0\}$ is a singleton since it has only one element 0.

(ii) The set of even prime numbers is the set $\{2\}$ which is a singleton.

(iii) $\{x \mid x$ is an integer and $-1 < x < 1\} = \{0\}$ is a singleton.

EQUAL SETS

Two sets A and B are said to be *equal* if they have the same elements and we write $A = B$. Thus, $A = B$ iff every element of A is an element of B and every element of B is an element of A.

In symbols, $A = B$ iff $x \in A \Rightarrow x \in B$ and $x \in B \Rightarrow x \in A$. To indicate that two sets A and B are not equal, we shall write $A \neq B$.

CAUTION

A set does not change if one or more elements of the set are repeated. For example, the sets A = {2, 5, 7} and B = {2, 5, 5, 7} are equal, since each element of A is in B and vice-versa. That is why we generally do not repeat any element in describing a set.

Illustration 6.

(i) If $A = \{2, 3, 4\}$ and $B = \{x \mid 1 < x < 5, x \in N\}$ then $A = B$.

(ii) If A = the set of letters in the word WOLF and B = the set of letters in the word FOLLOW, then $A = B$, each = {W, O, L, F} remembering that in a set the repetition of elements is meaningless and order of elements is immaterial.

Example 1. *Which of the following sets are empty sets?*

(i) A = *set of odd natural numbers divisible by* 2;

(ii) B = *set of even prime numbers*;

(iii) $C = \{x : x$ *is a natural number*, $x < 5$ *and* $x > 7\}$;

(iv) $D = \{x : x$ *is a point common to any two parallel lines*$\}$.

Solution:

(i) As no odd natural number is divisible by 2, so the set A is empty.

(ii) Since 2 is an even prime number, $\therefore B = \{2\}$. So B is not an empty set.

(iii) There is no natural number which is less than 5 as well as greater than 7, therefore, C is an empty set.

(iv) As there is no point common to parallel lines, so the set D is an empty set.

Example 2. *State which of the following sets are finite and which are infinite:*

(i) $A = \{x : x \in Z$ *and* $x^2 - 2x - 3 = 0\}$;

(ii) B = *the set of natural numbers which are divisible by* 2;

(iii) C = *the set of lines passing through a point*;

(iv) $D = \{x : x \in Z$ *and* $x^2 = 25\}$;

(v) E = *the set of points common to two given parallel lines*;
(vi) $F = \{x : x \in Z \text{ and } x > -5\}$;
(vii) $G = \{x : x \in Z \text{ and } x^2 \text{ is even}\}$.

Solution:

(i) $A = \{x : x \in Z \text{ and } x^2 - 2x - 3 = 0\} = \{3, -1\}$
$\therefore$ A is a finite set.

(ii) B = the set of natural numbers divisible by 2 = $\{2, 4, 6, 8, 10, ...\}$
$\therefore$ B is an infinite set.

(iii) Since infinite number of lines pass through a point, therefore, C is an infinite set.

(iv) $D = \{x : x \in Z \text{ and } x^2 = 25\} = \{5, -5\}$. Clearly, D is a finite set.

(v) Since, the set of points common to two given parallel lines is empty, $\therefore$ the set E is a finite set.

(vi) $F = \{x : x \in Z \text{ and } x > -5\} = \{-4, -3, -2, ...\}$. Clearly, F is an infinite set.

(vii) $G = \{x : x \in Z \text{ and } x^2 \text{ is even}\} = \{..., -6, -4, -2, 0, 2, 4, 6, ...\}$. Clearly, G is an infinite set.

Example 3. *Are the following sets equal?*

A = {*x* : *x is a letter in the word* REAP}
B = {*x* : *x is a letter in the word* PAPER}
C = {*x* : *x is a letter in the word* RAPE}.

Solution: A = {x : x is a letter in the word REAP} = {R, E, A, P},
B = {x : x is a letter in the word PAPER}
= {P, A, P, E, R} = {P, A, E, R} = {R, E, A, P}
C = {x : x is a letter in the word RAPE} = {R, A, P, E} = {R, E, A, P}

Thus, {R, E, A, P} = $A = B = C$ or $A = B = C$.

Hence the given sets are equal.

Example 4. *Show that the following sets are equal:*

$$A = \{2, 1\},\ B = \{2, 1, 1, 2, 1, 2\}$$
$$C = \{x : x^2 - 3x + 2 = 0\}$$

Solution: Here $A = \{2, 1\}$

$B = \{2, 1, 1, 2, 1, 2\} = \{1, 2\}$

and $C = \{x : x^2 - 3x + 2 = 0\} = \{x : (x - 1)(x - 2) = 0\} = \{1, 2\}$

From these we observe that the sets A, B, C contain the same elements and hence these are equal.

Example 5. *Which of the following pairs of sets are equal?*

(i) $A = \{1, 3, 3, 1\}, B = \{1, 4\}$; (ii) $A = \{x : x + 2 = 2\}, B = \{0\}$;
(iii) $A = \{1, 3, 4, 4\}, B = \{3, 1, 4\}$; (iv) $A = \left\{1, \frac{1}{2}, \frac{1}{3}, ...\right\}, B = \left\{\frac{1}{n} : n \in N\right\}$;
(v) $A = \{x : x \in W\}, B = \{x : x \in N\}$;
(vi) A = *the set of letters in the word* MATHEMATICS,
B = *the set of letters in the word* MATCHES.

Solution:

(i) $A = \{1, 3\}, B = \{1, 4\}$
A and B have different elements, $\therefore A \neq B$.

(ii) $A = \{x : x + 2 = 2\} = \{0\}, B = \{0\}$

A and B have same elements, $\therefore A = B$.

(iii) $A = \{1, 3, 4, 4,\} = \{1, 3, 4\}, B = \{3, 1, 4\}$.

A and B have same elements, $\therefore A = B$.

(iv) $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\} = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\right\}, B = \left\{\frac{1}{n} : n \in N\right\} = \left\{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$

A and B have same elements, $\therefore A = B$.

(v) $A = \{x : x \in W\} = \{0, 1, 2, 3, \ldots\}$

$B = \{x : x \in N\} = \{1, 2, 3, \ldots\}$

A and B differ in one element, $\therefore A \neq B$.

(vi) A = the set of letters in the word MATHEMATICS

= {M, A, T, H, E, I, C, S}

B = the set of letters in the word MATCHES = {M, A, T, C, H, E, S}

A and B have different elements, $\therefore A \neq B$.

Example 6. *Find the pairs of equal sets, from the following sets, if any, giving reasons.*

$A = \{0\}, B = \{x : x > 15 \text{ and } x < 5\}, C = \{x : x - 5 = 0\}, D = \{x : x^2 = 25\}$

$E = \{x : x \text{ is an integral positive root of the equation } x^2 - 2x - 15 = 0\}$

Solution: We have

$A = \{0\}, B = \{x : x > 15 \text{ and } x < 5\} = \phi, \quad C = \{x : x - 5 = 0\} = \{5\},$

$D = \{x : x^2 = 25\} = \{-5, 5\}$ and $E = \{5\}$. Clearly, $C = E$.

EXERCISE 1.2

1. Which of the following sets are empty?

(i) $A = \{x : x \in N \text{ and } x \leq 1\}$; (ii) $B = \{x : 3x + 1 = 0, x \in N\}$;

(iii) $C = \{x : 2 < x < 3, x \in N\}$; (iv) $D = \{x : x \text{ is an integer and } -1 < x < 1\}$;

(v) E = set of months of the year beginning with F;

(vi) F = set of days of the week beginning with J;

(vii) $G = \{x : x^2 - 3 = 0 \text{ and } x \text{ is rational}\}$;

(viii) $H = \{x : x + 5 = 0, x \in N\}$;

(ix) $I = \{x : 1 < x < 2, x \in N\}$;

(x) $J = \{x : x \in N, x \text{ is even}, x \text{ is odd}\}$.

2. Which of the following sets are finite and which are infinite?

(i) $\{x : x \text{ is a prime number}, x \text{ is even}\}$; (ii) Set of all rivers in India;

(iii) Set of all concentric circles;

(iv) $\{x : x \text{ is a multiple of 2}, x \text{ is an integer}\}$.

3. Which of the following sets are finite or infinite?

(i) The set of the months of a year; (ii) {1, 2, 3, ...}; (iii) {1, 2, 3, ... 90, 100};

(iv) The set of positive integers greater than 100;

(v) The set of prime numbers less than 99.

4. State whether each of the following sets is finite or infinite:

(i) The set of lines which are parallel to the x-axis;

(ii) The set of letters in the English alphabet;

(iii) The set of numbers which are multiplie of 5;

(iv) The set of animals living on earth;

(v) The set of circles passing through the origin (0, 0).

5. From the following sets select equal sets:
$A = \{2, 4, 8, 12\}$; $B = \{1, 2, 3, 4\}$; $C = \{4, 8, 12, 14\}$; $D = \{3, 1, 4, 2\}$;
$E = \{-1, 1\}$; $F = \{0, 9\}$; $G = \{1, -1\}$; $H = \{0, 1\}$.

6. From the sets given below, pair the equal sets:
$A = \{1, 2, 3, 4\}$; $B = \{p, q, r, s\}$; $C = \{1, 4, 9, 16\}$; $D = \{w, x, y, z\}$;
$E = \{16, 1, 9, 4\}$; $F = \{4, 2, 3, 1\}$; $G = \{r, p, q, s\}$; $H = \{z, w, w, y, x\}$;
$P = \{x : 2 < x < 3, x \in N\}$; $Q = \{x : x^2 + 1 = 0, x \in N\}$.

7. Are the following pairs of sets equal? Give reasons.
(i) $A = \{2, 3\}$, $B = \{x : x$ is a solution of $x^2 + 5x + 6 = 0\}$;
(ii) $A = \{x : x$ is a letter in the word FOLLOW$\}$,
$B = \{y : y$ is a letter in the word WOLF$\}$.

8. In the following, state whether $A = B$ or not:
(i) $A = \{a, b, c, d\}$, $B = \{d, c, b, a\}$;
(ii) $A = \{4, 8, 12, 16\}$, $B = \{8, 4\ 16, 18\}$;
(iii) $A = \{2, 4, 6, 8, 10\}$, $B = \{x : x$ is positive even integer and $x \le 10\}$;
(iv) $A = \{x : x$ is a multiple of $10\}$, $B = \{10, 15, 20, 25, 30, ...\}$.

9. State which of the following sets are infinite:
(i) $\{x : x \in N$ and x is a prime number$\}$;
(ii) $\{x : x$ is a quadrilateral on a plane$\}$;
(iii) $\{x : x \in N$ and $x^2 - 25 \le 0\}$;
(iv) $\{x : x \in N$ and x is a multiple of $3\}$;
(v) $\{x : x \in R$ and $0 < x < 5\}$;
(vi) $\{x : x$ is a factor of $2100\}$;
(vii) $\{x : x$ is a house in America$\}$.

10. State in which of the following cases, $A = B$:
(i) $A = \{12, 14, 16\}$, $B = \{16, 18, 20\}$;
(ii) $A = \{2, 4, 6, 8\}$, $B = \{4, 8, 6, 2\}$;
(iii) $A = \phi$, $B = \{\}$;
(iv) $A = \{x : x \in W$ and $x < 1\}$, $B = \phi$;
(v) $A = \{x : x$ is a day of the week beginning with $S\}$, $B = \{$Sunday$\}$;
(vi) $A = \{x : x$ is a letter of the word TOP$\}$, $B = \{x : x$ is a letter of the word POT$\}$.
(vii) $A = \{x : x \in Z$ and $x^2 < 8\}$, $B = \{x : x \in R$ and $x^2 - 4x + 3 = 0\}$.

11. Which of the following sets are equal?
(i) The set consisting of the alphabets A, E, S and T.
(ii) The set of letters in the word TEASE.
(iii) $\{x : x$ is a letter in the word SEAT$\}$
(iv) $\{x : x$ is a letter in the word EAST$\}$
(v) The set of letters in the word ASSETS.

Answers

1. B, C, F, G, H, I and J.

2. (i) Finite; (ii) Finite; (iii) Infinite; (iv) Infinite.

3. (i) Finite; (ii) Infinite; (iii) Finite; (iv) Infinite;
(v) Finite; (vi) Infinite; (vii) Infinite.

4. (i) Infinite; (ii) Finite; (iii) Infinite; (iv) Finite; (v) Infinite.

5. B and D; E and G. **6.** A and F; B and G; C and E; D and H; P and Q.

7. (i) No. Since $x^2 + 5x + 6 \Rightarrow 0 = (x + 2)(x + 3) = 0 \Rightarrow x = -2$ or $x = 3$
$\therefore$ Solution set $= \{-2, -3\}$.
Clearly $A \neq B$.

(ii) Yes. $A = \{F, O, L, W\}$ and $B = \{W, O, L, F\}$. Since every element of A is in B and every element of B is in A, therefore, $A = B$.

8. (i) Yes; (ii) No; (iii) Yes; (iv) No.

9. (i), (ii), (iv), (v) and (vii) are infinite sets.

10. (ii), (iii), and (vi) are equal sets.

11. All.

HINTS AND SOLUTIONS

3. (i) It is a finite set, as there are 12 members of the set which are the months of the year.
(ii) It is an infinite set since there are infinite number of natural numbers.
(iii) It is a finite set as it contains first 100 natural numbers.
(iv) It is an infinite set since there are infinite number of positive integers viz. 101, 102, 103, ... greater than 100.
(v) It is a finite set because the set is $\{2, 3, 5, 7 ..., 97\}$.

4. (i) Infinite. Infinite lines can be drawn parallel to x-axis.
(ii) Finite. It is a set of 26 letters.
(iii) Infinite. $\{5, 10, 15, ...\}$
(iv) Finite. There are finite number of animals living on earth.
(v) Infinite. Infinite number of circles can be drawn passing through the origin.

10. (vii) $A = \{-2, -1, 0, 1, 2\}$ and $B = \{1, 3\}$.
Since $0 \in A$, but $0 \notin B$. So, $A \neq B$.

SUBSET OF A SET

If A and B are any two sets, then B is called a *subset* of A if every element of B is also an element of A.

Symbolically, we write it as, $B \subseteq A$ or $A \supseteq B$.

(i) $B \subseteq A$ is read as B is contained in A or B is a subset of A.

(ii) $A \supseteq B$ is read as A contains B or A is *super-set* of B.

Illustration 1.

(i) The set $A = \{2, 4, 6\}$ is a subset of $B = \{1, 2, 3, 4, 5, 6\}$, since each number 2, 4 and 6 belonging to A, also belongs to B.

(ii) The set $A = \{1, 3, 5\}$ is not a subset of $B = \{1, 2, 3, 4\}$ since $5 \in A$, but $5 \notin B$.

(iii) The set of real numbers is a subset of the set of complex numbers. The set of rational numbers is a subset of the set of real numbers. The set of integers is a subset of the set of rational numbers. Finally, the set of natural numbers is a subset of the set of integers. Symbolically,

$$N \subseteq Z \subseteq Q \subseteq R \subseteq C$$

Trick(s) for Problem Solving

- If we are to prove that $A \subseteq B$, then we should prove that $x \in A \Rightarrow x \in B$. Symbolically,

$$A \subseteq B \text{ if and only if } x \in A \Rightarrow x \in B$$

- If we are to prove that $A \not\subseteq B$, then we should prove that there exists atleast one element x such that $x \in A$ but $x \notin B$. Symbolically,

$$A \not\subseteq B \text{ if and only if there exists } x \in A \text{ such that } x \notin B$$

Proper Subsets of a Set

A set B is said to be a *proper subset* of the set A if every element of set B is an element of A, whereas every element of A is not an element of B.

We write it as $B \subset A$ and read it as 'B is a proper subset of A'.

Thus, B is a proper subset of A if every element of B is an element of A, and there is at least one element in A which is not in B.

Illustration 2.

(i) If $A = \{1, 2, 5\}$ and $B = \{1, 2, 3, 4, 5\}$. Then A is a proper subset of B.

(ii) The set N of all natural numbers is a proper subset of the set Z of all integers because every natural number is an integer, i.e., $N \subseteq Z$, but every integer need not be a natural number, i.e., $N \neq Z$.

Trick(s) for Problem Solving

If we are to prove that $B \subset A$, then we should prove that $B \subseteq A$ and there exists an element of A which is not in B. Symbolically, $B \subset A$ if and only if $B \subseteq A$ and there exists $x \in A$ such that $x \notin B$.

POWER SET

Elements of a set can also be some sets. Such sets are called *set of sets*. For example, the set $\{\phi, \{1\}, \{2\}, \{3, 4\}\}$ is a set whose elements are the sets ϕ, $\{1\}$, $\{2\}$, $\{3, 4\}$.

The set of all the subsets of a given set A is called the *power set* of A and is denoted by $P(A)$.

Since the empty set and the set A itself are subsets of A and are, therefore, elements of $P(A)$. Thus, the powerset of a given set is always non-empty.

Illustration 3.

(i) If $A = \{a\}$, then $P(\text{A}) = \{\phi, A\}$.

(ii) If $B = \{2, 5\}$, then $P(\text{B}) = \{\phi, \{2\}, \{5\}, B\}$.

(iii) If $S = \{a, b, c\}$, then $P(S) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, S\}$.

IMPORTANT

1. Every set is subset of itself.
2. Empty set is the subset of every set.
3. If a set has n elements, then the number of its subsets is 2^n.

UNIVERSAL SET

If in any discussion on the set theory all the given sets are subsets of a set U, then the set U is called the *universal set*.

Illustration 4. Let $A = \{2, 4, 6\}$, $B = \{1, 3, 5\}$, $C = \{3, 5, 7, 11\}$, $D = \{2, 4, 8, 16\}$ and $U = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 16\}$ be the given sets. Here the sets A, B, C and D are subsets of the set U. Hence U can be taken as the *universal set*.

Intervals as Subset of *R*

The set of all numbers lying between two given real numbers is called an *interval* in R.

Let a and b be any two real numbers such that $a < b$, then we define the following types of intervals:

(i) Closed interval [a, b]

Fig. 1.1

(a, b) = closed interval from a to b
= $\{x : x \in R;\ a \leq x \leq b\}$
= set of all real numbers lying between a and b including the end points a and b.

Clearly, it is an infinite subset of R.

(ii) Open interval (a, b) or]a, b[

Fig. 1.2

= open interval from a to b
= $\{x : x \in R;\ a < x < b\}$
= set of all real numbers lying between a and b, excluding the end points a and b.
This is an infinite subset of R.

(iii) Closed-open interval [a, b) and open-closed interval (a, b]

Fig. 1.3

$[a, b) = \{x : x \in R;\ a \leq x < b\}$

Fig. 1.4

$(a, b] = \{x : x \in R;\ a < x \leq b\}$

(iv) Real number set R as an open interval

We introduce two special numbers $-\infty$ and $+\infty$, where

$-\infty$ = a number less than any real number,
$+\infty$ = a number greater than any real number.

$$-\infty < x \text{ for all } x \in R, \text{ and } x < \infty \text{ for all } x \in R.$$

Hence the set R can be thought of as the open interval $(-\infty, \infty)$, so that

$$R = (-\infty, \infty) = \{x : x \in R;\ -\infty < x < \infty\}$$

Also, the infinite intervals in R can be given by

$$(-\infty, a), (a + \infty), (-\infty, a], [a + \infty)$$

Clearly all of these are infinite subsets of R.

CAUTION

The numbers $+\infty$ and $-\infty$ do not follow the ordinary rules of arithmetic.

Illustration 5. $\infty - \infty \neq 0,\ 0 \times \infty \neq 0,\ \dfrac{0}{0} \neq 1,\ \dfrac{\infty}{\infty} \neq 1,\ \infty + \infty \neq 2\infty,\ 1^{\infty} \neq 1,\ 0^0 \neq 1$, etc.

Example 1. *Replace * by ⊆ or ⊄ to make the statement correct in the following:*

(i) $\{1, 7, 8\} * \{6, 7, 8, 9\}$; (ii) $\{1, 2, 3\} * \{1, 2, 3, 4\}$;

(iii) $\{x : x \in N$ *and* $x \geq 5\} * \{5, 6, 7, 8, 9\}$;

(iv) $\{x : x$ *is an equilateral triangle*$\} * \{x : x$ *is an isosceles triangle*$\}$.

Solution: (i) $1 \in \{1, 7, 8\}$, but $1 \notin \{6, 7, 8, 9\}$

$\therefore \{1, 7, 8\} \not\subseteq \{6, 7, 8, 9\}$

(ii) Since every member of $\{1, 2, 3\}$ is also a member of $\{1, 2, 3, 4\}$,

$\therefore \{1, 2, 3\} \subseteq \{1, 2, 3, 4\}$

(iii) Since $\{x : x \in N \text{ and } x \geq 5\} = \{5, 6, 7, 8, 9, ...\}$,

$\therefore$ every member of $\{5, 6, 7, 8, 9, ...\}$ is not a member of $\{5, 6, 7, 8, 9\}$.

Thus, $\{x : x \in N \text{ and } x \geq 5\} \not\subseteq \{5, 6, 7, 8, 9\}$

(iv) $\{x : x$ is an equilateral triangle$\}$ = The set of all triangles having three sides equal and $\{x : x$ is an isosceles triangle$\}$ = The set of all triangles having two sides equal.

But a triangle having three sides equal already has two sides equal,

$\therefore$ every element of $\{x : x$ is an equilateral triangle$\}$ is an element of $\{x : x$ is an isosceles triangle$\}$,

$\therefore$ $\{x : x$ is an equilateral triangle$\} \subseteq \{x : x$ is an isosceles triangle$\}$

Example 2. *Let $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3\}$ and $C = \{2, 4\}$.*
Find all sets X such that:
(i) *$X \subset B$ and $X \subset C$;* (ii) *$X \subset A$ and $X \not\subset B$.*

Solution: (i) We have,

$$P(B) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

and $$P(C) = \{\phi, \{2\}, \{4\}, \{2, 4\}\}$$

Now, $X \subset B$ and $X \subset C \Rightarrow X \in P(B)$ and $X \in P(C) \Rightarrow X = \phi, \{2\}$

(ii) We have, $X \subset A$ and $X \not\subset B$

$\Rightarrow$ X is a subset of A, but X is not a subset of B.

$\Rightarrow$ $X \in P(A)$ but $x \notin P(B)$

$\Rightarrow$ $X = \{4\}, \{1, 2, 4\}, \{2, 3, 4\}$ $\{1, 3, 4\}, \{1, 4\}$ $\{2, 4\}, \{3, 4\}, \{1, 2, 3, 4\}$

Example 3. *Let $A = \{a, b, c, d\}$, $B = \{a, b, c\}$ and $C = \{b, d\}$. Find all sets X satisfying each pair of conditions:*
(i) *$X \subset B$ and $X \not\subset C$;* (ii) *$X \subset B$, $X \neq B$ and $X \not\subset C$;*
(iii) *$X \subset A$, $X \subset B$ and $X \subset C$.*

Solution: (i) We have, $X \subset B$ and $X \not\subset C$

$\Rightarrow$ X is a subset of B, but X is not a subset of C.

$\Rightarrow$ $X \in P(B)$, but $x \notin P(C)$

$\Rightarrow$ $X = \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$

(ii) We have, $X \subset B$, $X \neq B$ and $X \not\subset C$

$\Rightarrow$ X is a subset of B other than B itself and X is not a subset of C.

$\Rightarrow$ $X \in P(B)$, $X \notin P(C)$ and $X \neq B$

$\Rightarrow$ $X = \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{b, \text{c}\}$

(iii) We have, $X \subset A$, $X \subset B$ and $X \subset C$

$\Rightarrow$ $X \in P(A)$, $X \in P(B)$ and $X \in P(C)$

$\Rightarrow$ X is a subset of A, B and C.

$\Rightarrow$ $X = \phi, \{b\}$

Example 4. ***List all the subsets of $\{a, b, c\}$.***

Solution: The subsets of $\{a, b, c\}$ are $\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}$.

Example 5. *List the proper subsets of*

$$A = \{1, 2, 3\}$$

Solution: The proper subsets of A are $\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}$.

Example 6. *Given A is the set of letters in the word* MOON. *Find the power set of A.*

Solution: Power set of $A = P(A) = \{\phi, \{m\}, \{o\}, \{n\}, \{m, o\}, \{m, n\}$ $\{o, n\}, \{m, o, n\}\}$

Example 7. *Write down the power set of the following sets:* (i) $\{1, 2, 3\}$; (ii) $\{0\}$.

Solution: (i) Let $A = \{1, 2, 3\}$. The possible subsets of this set A are:

$$\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}$$

So, the power set of the given set A is:

$$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$$

(ii) Let $A = \{0\}$. The possible subsets of this set A are ϕ, $\{0\}$, so the power set of the given set A is:

$$P(A) = \{\phi, \{0\}\}$$

Example 8. *Find the power set of* $\{\{a, b\}, c\}$.

Solution: Let $A = \{\{a, b\}, c\}$. Since A contains two elements $\{a, b\}$, c, hence $P(A)$ will contain $2^2 = 4$ elements.

The elements of $P(A)$ are ϕ, A, $\{a, b\}$, $\{c\}$.

Example 9. *Write down the power set of set* $A = \{1, 2, \{3, 4\}\}$.

Solution: The elements of the given set A are 1, 2 and the set $\{3, 4\}$. The subsets of set A are:

$$\phi, \{1\}, \{2\}, \{3, 4\}, \{1, 2\}, \{1, \{3, 4\}\}, \{2, \{3, 4\}\}, \{1, 2, \{3, 4\}\}, \text{ i.e., } A.$$

$$\therefore \quad P(A) = \{\phi, \{1\}, \{2\}, \{3, 4\}, \{1, 2\}, \{1, \{3, 4\}\}, \{2, \{3, 4\}\}, A\}$$

Example 10 *Let B be a subset of a set A and let $P(A : B) = \{X \in P(A) \mid X \supset B\}$*

(i) *Show that $P(A : \phi) = P(A)$.*

(ii) If $A = \{a, b, c, d\}$ *and* $B = \{a, b\}$. *List all the members of the set $P(A : B)$.*

Solution: (i) We have,

$$P(A : B) = \{X \in P(A) \mid B \subset X\}$$

= Set of all those subsets of A which contain B

$\therefore P(A : \phi)$ = Set of all those subsets of A which contain ϕ

= Set of all subsets of set A

$= P(A)$

(ii) We have,

$$A = \{a, b, c, d\} \text{ and } B = \{a, b\}$$

$\therefore P(A : B)$ = Set of all those subsets of set A which contain B

$= \{a, b, c\}, \{a, b, d\}, \{a, b, c, d\}$

EXERCISE 1.3

1. Let $E = \{a, b\}$. Which one of the following statements are correct or incorrect?

   (i) $\{a\} \in E$; (ii) $\{b\} \subset E$; (iii) $\phi \in E$;
   (iv) $\phi \subset E$; (v) $\{a, b\} \subset E$; (vi) $E \subset \{a, b, c\}$.

2. Write the following in set notation:

   (i) B is a subset of A; (ii) C is not a subset of A;
   (iii) A is a super set of B; (iv) the set A belongs to the set B;
   (v) the set F is contained in the set B; (vi) T is a set having ϕ as a member.

3. Let $A = \{1, \{a, b\}\}$. Which one of the following statements are correct or incorrect?
(i) $1 \in A$; (ii) $a \in A$; (iii) $\{a, b\} \subset A$;
(iv) $\{1, a, b\} \subset A$; (v) $\{a, b\} \in A$; (vi) $\{1\} \subset A$.

4. Make correct statements by filling in the symbols $\subset$ or $\not\subset$ in the blank spaces.
(i) $\{2, 3, 4\}$ —— $\{1, 2, 3, 4, 5\}$; (ii) $\{a, b, c\}$ —— $\{b, c, d\}$;
(iii) $\{x : x$ is a student of Class XI of your school$\}$ —— $\{x : x$ is a student of your school$\}$;
(iv) $\{x : x$ is a circle in the plane$\}$ —— $\{x : x$ is a circle with radius 1$\}$;
(v) $\{x : x$ is a triangle in the plane$\}$ —— $\{x : x$ is a rectangle in the plane$\}$;
(vi) $\{x : x$ is an equilateral triangle in the plane$\}$ —— $\{x : x$ is a triangle in the plane$\}$;
(vii) $\{x : x$ is an even natural number$\}$ —— $\{x : x$ is an integer$\}$.

5. Examine whether the following statements are true or false:
(i) $\{a, b\} \not\subset \{b, c, a\}$; (ii) $\{a, e\} \subset \{x : x$ is a vowel in the English alphabet$\}$;
(iii) $\{1, 3, 5\} \subset \{1, 3, 5\}$; (iv) $\{a\} \subset \{a, b, c\}$. (v) $\{a\} \in \{a, b, c\}$.
(vi) $\{x : x$ is an even natural number which divides 6$\} \subset \{x : x$ is a natural number which divides 36$\}$.

6. Let $A = \{\phi, \{\phi\}, 1, \{1, \phi\}, 7\}$. Which of the following are true?
(i) $\phi \in A$; (ii) $\{\phi\} \in A$; (iii) $\{1\} \in A$;
(iv) $\{7, \phi\} \subset A$; (v) $7 \subset A$; (vi) $\{7, \{1\}\} \not\subset A$;
(vii) $\{\{7\}, \{1\}\} \not\subset A$; (viii) $\{\phi, \{\phi\}, \{1, \phi\}\} \subset A$; (ix) $\{\{\phi\}\} \subset A$.

7. Let $A = \{1, 2, \{3, 4\}, 5\}$. Which of the following statements are false and why?
(i) $\{3, 4\} \subset A$; (ii) $\{3, 4\} \in A$; (iii) $\{\{3, 4\}\} \subset A$;
(iv) $1 \in A$; (v) $1 \subset A$; (vi) $\{1, 2, 5\} \subset A$;
(vii) $\{1, 2, 5\} \in A$; (viii) $\{1, 2, 3\} \subset A$; (ix) $\phi \in A$; (x) $\{\phi\} \subset A$.

8. Let A, B and C be three sets. If $A \subset B$ and $B \in C$, is it true that $A \in C$? If not give an example.

9. Let $A = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7, 8\}\}$. Determine which of the following is true or false:
(i) $1 \in A$; (ii) $\{1, 2, 3\} \subset A$; (iii) $\{6, 7, 8\} \in A$;
(iv) $\{\{4, 5\}\} \subset A$; (v) $\phi \in A$; (vi) $\phi \subset A$.

10. (i) How many elements has $P(A)$ if $A = \phi$?
(ii) Prove that $A \subset \phi \Rightarrow A = \phi$.

11. Write the following as intervals:
(i) $\{x : x \in \mathbf{R}, -4 < x \le 6\}$ (ii) $\{x : x \in R, -12 < x < -10\}$
(iii) $\{x : x \in \mathbf{R}, 0 \le x < 7\}$ (iv) $\{x : x \in \mathbf{R}, 3 \le x \le 4\}$

12. Write down all the subsets of the following sets:
(i) $\{a\}$; (ii) $\{a, b\}$; (iii) $\{1, 2, 3\}$; (iv) ϕ.

13. Write the following intervals in set-builder form:
(i) $(-3, 0)$ (ii) $[6, 12]$; (iii) $(6, 12]$; (iv) $[-23, 5)$

14. Let $A = \{1, 3, 5\}$ and $B = \{x : x$ is an odd natural number $< 6\}$.
Is $A \subset B$? Is $B \subset A$? Is $A = B$?

15. Let $A = \{1, 2, \{3, 4\}, 5\}$.
Which of the following statements are true or false and why?
(i) $\{3, 4\} \subset A$; (ii) $\{3, 4\} \in A$; (iii) $\{\{3, 4\}\} \not\subset A$;
(iv) $\{1, 2, 5\} \subset A$; (v) $\{1, 3, 5\} \subset A$.

16. Write down the power sets of the following:
(i) $A = \{8, 9\}$; (ii) $B = \{0, 1, 2\}$ (iii) $C = \{1, \{2\}\}$.

17. Decide, among the following sets, which are the subsets of which?
$A = \{$all real numbers satisfying $x^2 - 8x + 12 = 0\}$, $B = \{2, 4, 6\}$,
$C = \{2, 4, 6, 8, ...\}$, $D = \{6\}$.

18. In each of the following, determine whether the statement is true or false. If it is true, prove it. If it is false, give an example.

(i) If $x \in A$ and $A \in B$, then $x \in B$; (ii) If $A \subset B$ and $B \in C$, then $A \in C$;
(iii) If $A \subset B$ and $B \subset C$, then $A \subset C$; (iv) If $A \not\subset B$ and $B \not\subset C$, then $A \not\subset C$;
(v) if $x \in A$ and $A \not\subset B$, then $x \in B$; (vi) If $A \subset B$ and $x \notin B$, then $x \notin A$.

19. State whether each of following statements is true or false for the sets A, B and C where: $A = \{x \mid x$ is a letter of the word BOWL$\}$, $B = \{x \mid x$ is a letter of the word ELBOW$\}$ $C = \{x \mid x$ is a letter of the word BELLOW$\}$

(i) $A \subset B$; (ii) $B \supset C$; (iii) $B = C$
(iv) A is a proper subset of B; (v) B is a proper subset of C.

20. Let $A = \{x \mid x$ is a letter in the word 'GEORGE CANTOR'$\}$ and $B = \{x \mid x$ is a vowel in the word 'GEORGE CANTOR'$\}$, then

(i) write the sets A, B in the tabular form; (ii) state $n(A)$ and $n(B)$;
(iii) write the number of proper subsets of A; (iv) write the power set of B.

21. Let $A = \phi$. Show that the power set $P[P(P(\phi))]$ has 4 elements.

22. Classify the following statements as True or False:

(i) Every subset of a finite set is finite;
(ii) Every subset of an infinite set is infinite;
(iii) Every subset of an infinite set is finite;
(iv) The power set of a given set is the set of all subsets of the set;
(v) A proper subset of a finite set is equivalent to the set itself;
(vi) For any two sets A and B either $A \subseteq B$ or $B \subseteq A$;
(vii) ϕ is a subset of $\{\phi\}$; (viii) $5 \subset \{2, 4, 5\}$;
(ix) $5 \in \{2, 4, 5\}$; (x) $5 \in \{2, 4, \{5\}\}$;
(xi) $\{0\} \subset \{0, 1, 2\}$; (xii) $\{0\} \in \{0, 1, 2\}$.

23. What universal set would you propose for each of the following:

(i) The set of right triangles;
(ii) The set of isosceles triangles.

24. Given the sets $A = \{1, 3, 5\}$, $B = \{2, 4, 6\}$, $C = \{0, 2, 4, 6, 8\}$.
Which of the following may be considered as universal set(s) for all the three sets A, B and C?

Answers

1. (i) Incorrect; (ii) Correct; (iii) Incorrect; (iv) Correct;
(v) Correct; (vi) Correct.

2. (i) $B \subset A$; (ii) $C \not\subset A$; (iii) $A \supset B$; (iv) $A \in B$;
(v) $F \subset B$; (vi) $\phi \in T$.

3. (i) Correct; (ii) Incorrect; (iii) Incorrect; (iv) Incorrect;
(v) Correct; (vi) Correct.

4. (i) $\subset$; (ii) $\not\subset$; (iii) $\subset$; (iv) $\not\subset$;
(v) $\not\subset$; (vi) $\subset$; (vii) $\subset$.

5. (i) False; (ii) True; (iii) False; (iv) True;
(v) False; (vi) True.

6. (i) True; (ii) True; (iii) False; (iv) True;
(v) False; (vi) True; (vii) True; (viii) True; (ix) True.

7. (i) False; (ii) True; (iii) True; (iv) True;
(v) False; (vi) True; (vii) False; (viii) False;
(ix) False; (x) False.

8. No; $A = \{1, 2\}$, $B = \{1, 2, 3\}$, $C = \{\{1, 2, 3\}\}$. Clearly, $A \subset B$ and $B \in C$, but $A \notin C$.

9. (i) False; (ii) False; (iii) True; (iv) True; (v) False; (vi) True.

10. (i) 1.

11. (i) $(-4, 6]$; (ii) $(-12, -10)$; (iii) $[0, 7)$; (iv) $[3, 4]$.

12. (i) ϕ, $\{a\}$; (ii) ϕ, $\{a\}$, $\{b\}$, $\{a, b\}$;
(iii) ϕ, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$. (iv) ϕ.

13. (i) $\{x : x \in R, -3 < x < 0\}$; (ii) $\{x : x \in R, 6 \le x \le 12\}$;
(iii) $(x : x \in R, 6 < x \le 12\}$; (iv) $\{x : x \in R, -23 \le x < 5\}$.

14. Yes; Yes; Yes.

15. (i) False, $\{3, 4\}$ is an element of A; (ii) True;
(iii) False, $\{\{3, 4\}\} \subset A$; (iv) True; (v) False, $3 \notin A$.

16. (i) $\{\phi, \{8\}, \{9\}, \{8, 9\}\}$;
(ii) $\{\phi, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$;
(iii) $\{\phi, \{1\}, \{\{2\}\}, \{1, \{2\}\}\}$.

17. $A \subset B$, $A \subset C$, $B \subset C$, $D \subset A$, $D \subset B$, $D \subset C$.

18. (i) False.
Let $A = \{1\}$, $B = \{\{1\}, 2\}$
Clearly, $1 \in A$ and $A \in B$, but $1 \notin B$
So, $x \in A$ and $A \in B$ need not imply that $A \in B$.

(ii) False.
Let $A = \{1\}$, $B = \{1, 2\}$ and $C = \{\{1, 2\}, 3\}$
Clearly, $A \subset B$ and $B \in C$, but $A \notin C$
Thus, $A \subset B$ and $B \in C$ need not imply that $A \in C$.

(iii) True.
Let $x \in A$. Then,
$$A \subset B \Rightarrow x \in B \Rightarrow x \in C$$
Thus, $x \in A \Rightarrow x \in C$ for all $x \in A \Rightarrow A \subset C$
Hence, $A \subset B$ and $B \subset C \Rightarrow A \subset C$.

(iv) False.
Let $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{1, 2, 5\}$
Then, $A \not\subset B$ and $B \not\subset C$, but $A \subset C$
Thus, $A \not\subset B$ and $B \not\subset C$ need not imply that $A \not\subset C$.

(v) False.
Let $A = \{1, 2\}$ and $B = \{2, 3, 4, 5\}$
Clearly $1 \in A$ and $A \not\subset B$, but $1 \not\subset B$
Thus, $x \in A$ and $A \not\subset B$ need not imply that $x \in B$.

(vi) True.
Let $A \subset B$. Then, clearly $x \in A \Rightarrow x \in B \Leftrightarrow x \notin B \Rightarrow x \notin A$.

19. (i) True; (ii) True; (iii) True; (iv) True; (v) False.

20. (i) $A = \{$G, E, O, R, C, A, N, T$\}$ and $B = \{$E, O, A$\}$.
(ii) $n(A) = 8$ and $n(B) = 3$.
(iii) The number of proper subsets of $A = 2^8 - 1 = 256 - 1 = 255$.
(iv) $P(B) = \{\phi$, (E$\}$, $\{$O$\}$, $\{$A$\}$, $\{$E, O$\}$, $\{$O, A$\}$, B$\}$.

22. (i), (iv), (vii), (ix), (xi) are true and (ii), (iii), (v), (vi), (viii), (x), (xii) are false.

23. The set of all the possible triangles.

24. (iii) i.e. {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

HINTS AND SOLUTIONS

5. (i) Since the elements of the set $\{a, b\}$ are also present in the set $\{b, c, a\}$, therefore, $\{a, b\} \not\subset \{b, c, a\}$ is false.

(ii) Vowels in the English alphabets are a, e, i, o, u.

$\therefore$ {a, e} $\subset$ $\{x : x$ is a vowel in the English alphabet$\}$ is true.

(iii) $\{1, 3, 5\} \subset \{1, 3, 5\}$ is false as they are equal sets. None is a proper subset of the other.

(iv) True as $\{a\}$ is contained in the set $\{a, b, c\}$.

(v) False because $a \in \{a, b, c\}$ and not $\{a\}$.

(vi) $\{x : x$ is an even natural number less than $6\} = \{2, 4\}$ and $\{x : x$ is a natural number which divides $36\} = \{1, 2, 3, 4, 6, 9, 12, 18, 36\}$. Since every element of the set $\{2, 4\}$ is contained in the set $\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$, therefore, the given statement is true.

10. (i) $P(A) = P(\phi) = \{\phi\}$ = a set containing 1 element.

(ii) Two sets A and B are said to be equal if and only if $A \subset B$ and $B \subset A$.

Now, $\phi \subset A$

and $A \subset \phi$ (Given)

$\therefore A = \phi$.

19. The given sets in the roster form are:

A = {B, O, W, L}, B = {E, L, B, O, W} and C = {B, E, L, O, W}.

21.
$$P(\phi) = \{\phi\}.$$
$$P(P(\phi)) = \{\phi, \{\phi\}\}$$
$$P[P(P(\phi))] = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$$

$\therefore$ $P[P(P(\phi))]$ has 4 elements.

VENN DIAGRAMS

In order to visualise and illustrate any property or theorem relating to universal sets, their subsets and certain operations on sets, Venn, a British Mathematician developed what are called *Venn diagrams*. He represented a universal set by interior of a rectangle and other sets or subsets by interiors of circles.

Illustrations of Certain Relationships Between Sets by Venn Diagrams

(i) If U be set of letters of English alphabets and A the set of vowels, then $A \subset U$. This relationship is illustrated by Fig. 1.1.

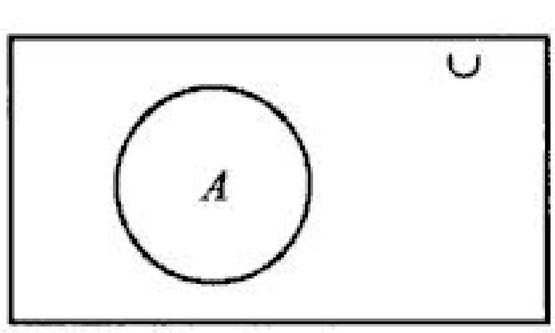

Fig. 1.5

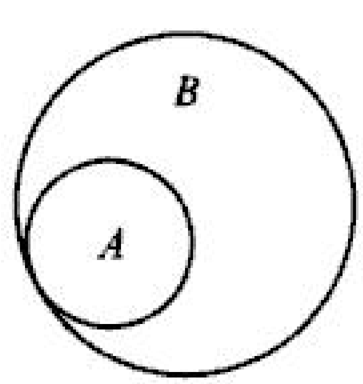

Fig. 1.6(a)

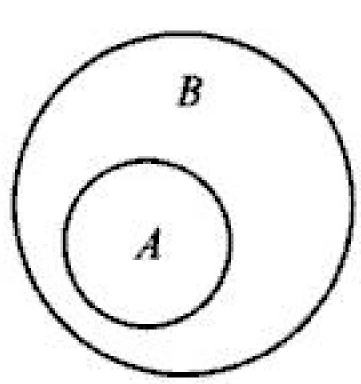

Fig. 1.6(b)

(ii) If $A \subset B$ and $A \neq B$, then A and B can be represented by either of the above diagrams [Fig. 1.6(a) and Fig. 1.6(b)].

(iii) If the sets A and B are not comparable, then neither of A or B is a subset of the other. This fact can be represented by either of the above diagrams [Fig. 1.7(a) and Fig. 1.7(b)].

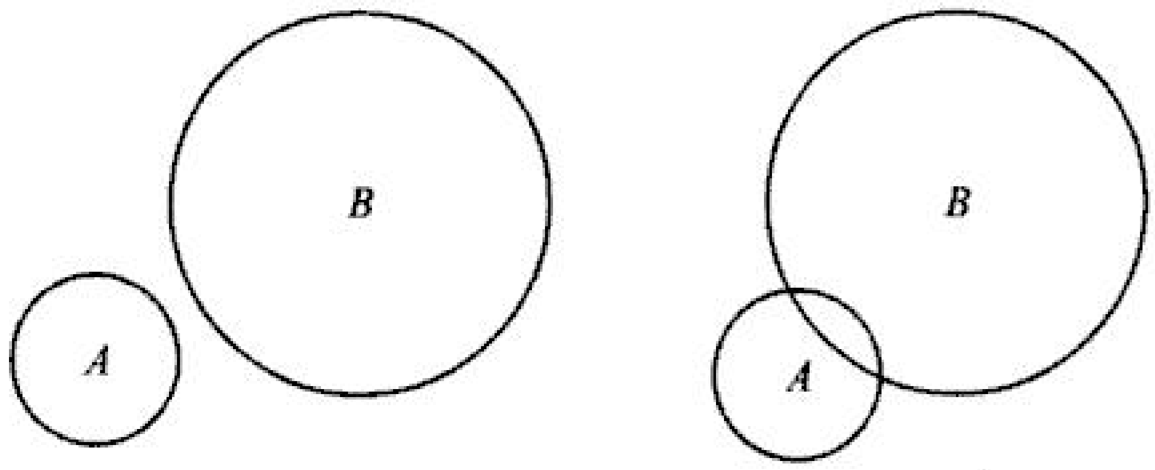

Fig. 1.7(a) Fig. 1.7(b)

(iv) If $A = \{1, 2, 3, 4\}$ and $B = \{5, 6, 7\}$, then A and B are disjoint. These can be illustrated by Venn-diagram given in Fig. 1.8.

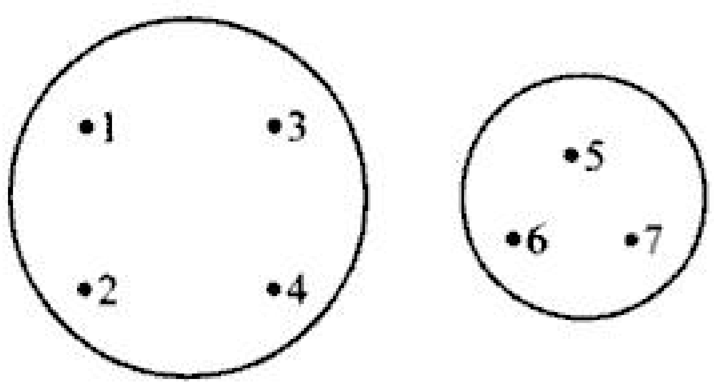

Fig. 1.8

COMPLEMENT OF A SET

Let $A \subset U$ (i.e., A is a proper subset of universal set U). Evidently, U consists of all the elements of A together with some elements which are not in A. Let us now constitute another set consisting of all the elements of U not in A. Naturally, it will form another proper subset of U. We call this subset '*the complement of the subset A in U*' and denote it by A' or by A^c, i.e., $A^c = \{x : x \in U, x \notin A\}$.

Thus, the complement of a given set is a set which contains all those members of the universal set that do not belong to the given set.

Representation of A′ by Venn-diagram

Let A be a subset of the universal set U. The shaded area in Fig. 1.9 represents the set A' which consists of those elements of U which are not in A.

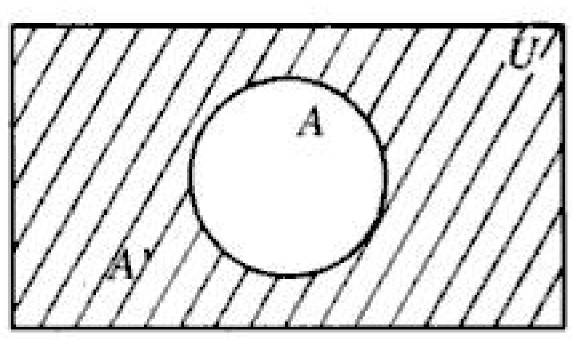

Fig. 1.9

Illustration 1.

(i) If the universal set is $\{a, b, c, d\}$ and $A = \{a, b, d\}$, then $A' = \{c\}$.

(ii) If universal set $U = \{1, 2, 3, 4, 5, 6\}$ and $A = \{2, 4, 6\}$, then $A' = \{1, 3, 5\}$.

(iii) If $U = N$ and $A = O$ (the set of odd natural numbers), then $A' = E$ (the set of even natural numbers).

(iv) If $U = I$, $A = N$, then $A' = \{0, -1, -2, -3 \ldots\}$

(v) If $U = \{1, 2, 3, 4\}$, $A = \{1, 2, 3, 4\}$, then $A' = \phi$.

Note: (i) Since $A \subset A$, therefore, $A' = \phi$.

(ii) $(A')' = A$, i.e., complement of the complement of a set is the set itself.

Proof. In order to prove $(A')' = A$, we have to prove that:

(a) $(A')' \subseteq A$

(b) $A \subseteq (A')'$

(a) Let x be any element of $(A')'$.

Then $\quad x \in (A')' \Rightarrow x \notin A' \Rightarrow x \in A$

$\therefore \quad (A')' \subseteq A$ ...(1)

(b) Let x be any element of A.

Then $\quad x \in A \Rightarrow x \notin A' \Rightarrow x \in (A')' \Rightarrow A \subseteq (A')'$ ...(2)

$\therefore$ From (1) and (2),

$(A')' = A$

OPERATIONS ON SETS

Union of Sets

Let A and B be two given sets. Then the union of A and B is the set of all those elements which belong to either A or B or both.

The union of A and B is denoted by $A \cup B$ and is read as '*A union B*'. The symbol $\cup$ stands for union. It is evident that union is '*either, or*' idea. Symbolically,

$$A \cup B = \{x : \text{either } x \in A \text{ or } x \in B\}$$

Note: The union set contains all the elements of A and B, except that the common elements of both A and B are exhibited only once.

Representation of $A \cup B$ by Venn diagram

Let A and B be any two sets contained in a universal set U. Then $A \cup B$ is indicated by the shaded area in Fig. 1.10.

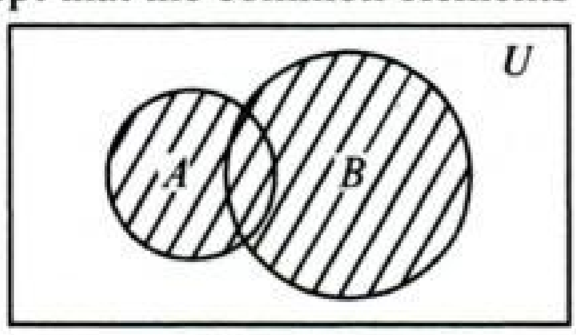

Fig. 1.10

Illustration 2.

(i) Let $A = \{1, 2, 3, 4\}$, $B = \{2, 3, 6, 7, 9\}$, then $A \cup B = \{1, 2, 3, 4, 6, 7, 9\}$.

(ii) If $A = O$ (set of odd natural numbers), $B = E$ (set of even natural numbers), then $A \cup B = N$.

(iii) If A is the set of rational numbers and B the set of irrational numbers, then $A \cup B = R$.

(iv) If $A = \{x : x^2 = 4, x \in I\} = \{2, -2\}$, $B = \{y : y^2 = 9, y \in I\} = \{3, -3\}$, then $A \cup B = \{-3, -2, 2, 3\}$.

(v) If $A = \{x : 1 < x < 5, x \in N\} = \{2, 3, 4\}$, $B = \{y : 3 < y < 7, y \in N\} = \{4, 5, 6\}$.

IMPORTANT

From the definition of the union of two sets A and B, it is clear that

- $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$
- $x \notin A \cup B \Leftrightarrow x \notin A$ and $x \notin B$
- $A \subseteq A \cup B$ and $B \subseteq A \cup B$
- $A \cup A' = \{x \in U : x \in A\} \cup \{x \in U : x \notin A\} = U$

Intersection of Sets

Let A and B be two given sets. Then the intersection of A and B is the set of elements which belong to both A and B. In other words, the intersection of A and B is the set of common members of A and B.

The intersection of A and B is denoted by $A \cap B$ and is read as '*A intersection B*'. The symbol $\cap$ stands for intersection.

It is evident that intersection is an '*and*' idea. Symbolically,

$$A \cap B = \{x : x \in A \text{ and } x \in B\}$$

IMPORTANT

From the definition of the intersection of two sets A and B, it is clear that

- $x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$
- $x \notin A \cap B \Leftrightarrow x \notin A$ or $x \notin B$
- $A \cap B \subseteq A$ and $A \cap B \subseteq B$
- $A \cap A' = \{x \in U : x \in A\} \cap \{x \in U : x \notin A\} = \phi$

Representation of A ∩ B by Venn diagram

Let A and B be any two sets contained in the universal set U. Then $A \cap B$ is indicated by the shaded area as shown in Fig. 1.11.

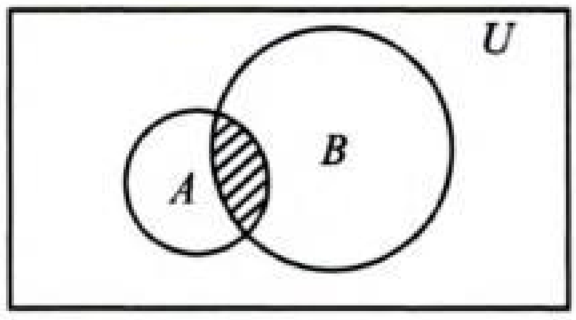

Fig. 1.11

Illustration 3.

(i) If $A = \{1, 2, 3, 6, 9, 18\}$ and $B = \{1, 2, 3, 4, 6, 8, 12, 24\}$, then $A \cap B = \{1, 2, 3, 6\}$.

(ii) If A is the set of odd natural numbers and B is the set of even natural numbers, then $A \cap B = \phi$. *[Intersection of two disjoint sets is empty set]*

(iii) If A and B are sets of points on two distinct concentric circles, then $A \cap B = \phi$.

(iv) If $A = \{x : 1 < x < 6, x \in N\} = \{2, 3, 4, 5\}$ and $B = \{y : 2 < y < 9, y \in N\} = \{3, 4, 5, 6, 7, 8\}$, then $A \cap B = \{3, 4, 5\}$

Disjoint sets

If $A \cap B = \phi$, then A and B are said to be disjoint sets. For example, let $A = \{2, 4, 6, 8\}$ and $B = \{1, 3, 5, 7\}$. Then, A and B are disjoint sets, because there is no element which is common to A and B. The disjoint sets can be represented by Venn diagram as shown in Fig. 1.12.

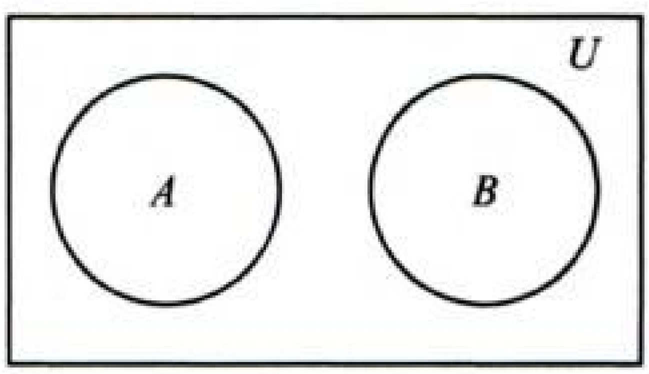

Fig. 1.12

Difference of Sets

Let A and B be two given sets. The difference of sets A and B is the set of elements which are in A but not in B. It is written as $A - B$ and read as 'A difference B'. Symbolically,

$$A - B = \{x : x \in A \text{ and } x \notin B\}$$

Similarly, $$B - A = \{x : x \in B \text{ and } x \notin A\}.$$

CAUTION

In general, $A - B \neq B - A$.

Illustration 4.

(i) If $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 7, 9\}$, then $A - B = \{1, 3\}$ and $B - A = \{7, 9\}$.
Hence $A - B \neq B - A$

(ii) If $A = \{12, 15, 17, 20, 21\}$, $B = \{12, 14, 16, 18, 21\}$ and $C = \{15, 17, 18, 22\}$, then

$$A - B = \{15, 17, 20\}$$
$$B - C = \{12, 14, 16, 21\}$$
$$C - A = \{18, 22\}$$
$$B - A = \{14, 16, 18\}$$
$$A - A = \phi.$$

Representation of A – B by Venn Diagrams

In the following four cases, shown in Figs. (1.13), (1.14), (1.15) and (1.16), $A - B$ is given by the shaded area.

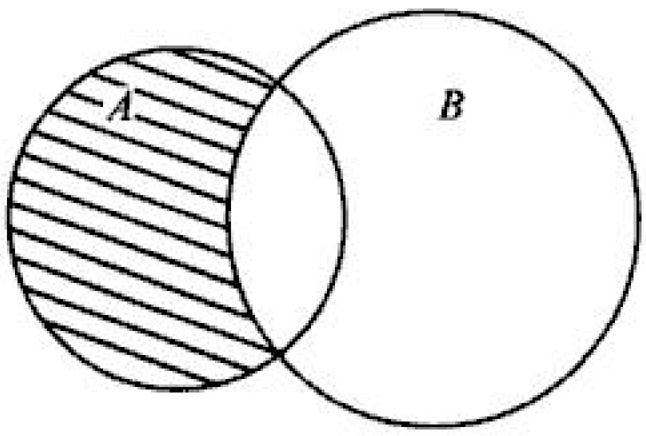

Fig. 1.13

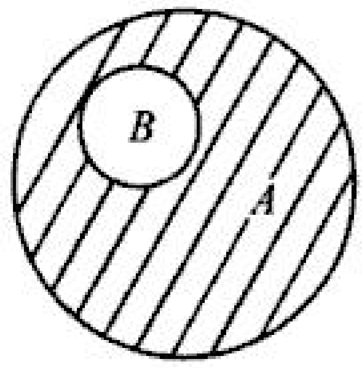

Fig. 1.14

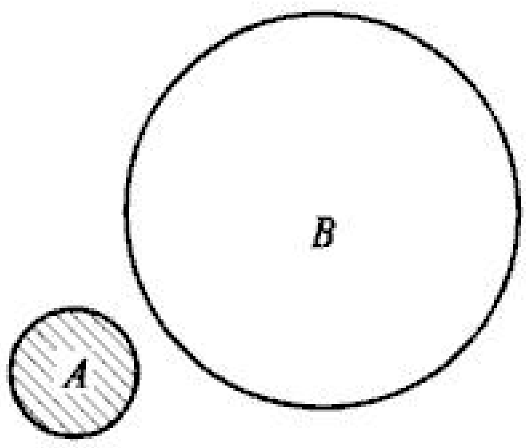

Fig. 1.15

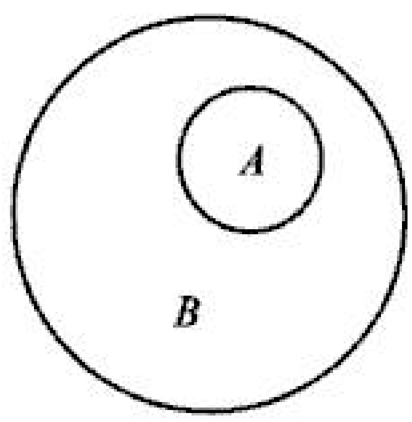

Fig. 1.16

Symmetric Difference of Two Sets

Let A and B be two given sets. The symmetric difference of sets A and B is the set $(A - B) \cup (B - A)$.

It is written as $A \Delta B$. Symbolically,

$A \Delta B = \{x : x \notin A \cup B\}$

In Fig. 1.17, the shaded part represents $A \Delta B$.

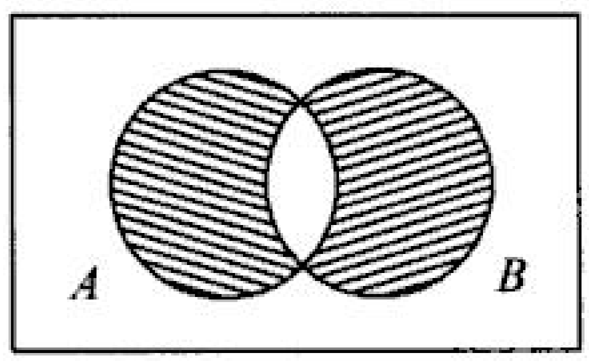

Fig. 1.17

LAWS OF ALGEBRA OF SETS

Operations on sets such as union and intersection satisfy various laws of algebra which are as follows:

Identity Laws

For any A, we have

(i) $A \cup \phi = A$ (ii) $A \cap U = A$

Proof. (i) $A \cup \phi = \{x : x \in A \text{ or } x \in \phi\} = \{x : x \in A\}$ (as $x \notin \phi$) $= A$

(ii) $A \cap U = \{x : x \in A \text{ and } x \in U\} = \{x : x \in A\} = A$

Idempotent Laws

For any set A, we have

(i) $A \cup A = A$ (ii) $A \cap A = A$

Proof. (i) $A \cup A = \{x : x \in A \text{ or } x \in A\} = \{x : x \in A\} = A$

(ii) $A \cap A = \{x : x \in A \text{ and } x \in A\} = \{x : x \in A\} = A$

Commutative Laws

For any two sets A and B, we have

(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

Proof. (i) Let x be an arbitrary element belonging to $A \cup B$.

Then $x \in A \cup B \Rightarrow x \in A$ or $x \in B \Rightarrow x \in B$ or $x \in A \Rightarrow x \in B \cup A$

$\therefore$ $A \cup B \subseteq B \cup A$

Similarly, $B \cup A \subseteq A \cup B$

$\therefore$ $A \cup B = B \cup A$

(ii) Let $x \in A \cap B$.

Then $x \in A$ and $x \in B \Rightarrow x \in B$ and $x \in A \Rightarrow x \in B \cap A$

$\therefore$ $A \cap B \subseteq B \cap A$

Similarly, $B \cap A \subseteq A \cap B$

$\therefore$ $A \cap B = B \cap A$

Associative Laws

For any three sets A, B and C, we have

(i) $(A \cup B) \cup C = A \cup (B \cup C)$ (ii) $(A \cap B) \cap C = A \cap (B \cap C)$

Proof. (i) Let $x \in (A \cup B) \cup C$.

Then $x \in (A \cup B)$ or $x \in C$ $\Rightarrow (x \in A$ or $x \in B)$ or $x \in C$

$\Rightarrow x \in A$ or $(x \in B$ or $x \in C)$

$\Rightarrow x \in A$ or $x \in (B \cup C) \Rightarrow x \in A \cup (B \cup C)$

$\therefore$ $(A \cup B) \cup C \subseteq A \cup (B \cup C)$

Similarly, $A \cup (B \cup C) \subseteq (A \cup B) \cup C$

$\therefore$ $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) Let $x \in (A \cap B) \cap C$.

Then $x \in (A \cap B)$ and $x \in C$ $\Rightarrow (x \in A$ and $x \in B)$ and $x \in C$

$\Rightarrow x \in A$ and $(x \in B$ and $x \in C)$

$\Rightarrow x \in A$ and $x \in (B \cap C)$

$\Rightarrow x \in A \cap (B \cap C)$

$\therefore$ $(A \cap B) \cap C \subseteq A \cap (B \cap C)$

Similarly, $A \cap (B \cap C) \subseteq (A \cap B) \cap C$

$\therefore$ $(A \cap B) \cap C = A \cap (B \cap C)$

Distributive Laws

For any three sets A, B and C, we have

(i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof. (i) Let $x \in A \cup (B \cap C)$.

Then $x \in A$ or $x \in (B \cap C)$ $\Rightarrow x \in A$ or $(x \in B$ and $x \in C)$

$\Rightarrow (x \in A$ or $x \in B)$ and $(x \in A$ or $x \in C)$

$\Rightarrow (x \in A \cup B)$ and $(x \in A \cup C)$

$\Rightarrow x \in (A \cup B) \cap (A \cup C)$

$\therefore$ $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$

Similarly, $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

$\therefore$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(ii) Let $x \in A \cap (B \cup C)$.

Then $x \in A$ and $x \in (B \cup C)$ $\Rightarrow x \in A$ and $(x \in B$ or $x \in C)$

$\Rightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$

$\Rightarrow x \in (A \cap B)$ or $x \in (A \cap C)$

$\Rightarrow x \in (A \cap B) \cup (A \cap C)$

$\therefore$ $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Similarly, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

$\therefore$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

De Morgan's Rule

For any two sets A and B,

(i) $(A \cup B)' = A' \cap B'$ and (ii) $(A \cap B)' = A' \cup B'$

Proof. (i) Let $x \in (A \cup B)'$.

Then $x \notin (A \cup B) \Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \in A' \cap B'$

$\therefore$ $(A \cup B)' \subseteq A' \cap B'$

Now, let $y \in A' \cap B'$.

Then $y \in A'$ and $y \in B' \Rightarrow y \notin A$ and $y \notin B$

$\Rightarrow y \notin A \cup B$

$\Rightarrow y \in (A \cup B)'$

$\therefore$ $A' \cap B' \subseteq (A \cup B)'$

Hence $(A \cup B)' = A' \cap B'$

(ii) Let $x \in (A \cap B)'$.

Then $x \notin (A \cap B) \Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \in A' \cup B'$

$\Rightarrow (A \cap B)' \subseteq A' \cup B'$

Also, let $y \in A' \cup B'$.

$\therefore$ $y \in A'$ or $y \in B'$ $\Rightarrow y \notin A$ or $y \notin B$

$\Rightarrow y \notin (A \cap B)$

$\Rightarrow y \in (A \cap B)'$

$\therefore$ $A' \cup B' \subseteq (A \cap B)'$

$\therefore$ $(A \cap B)' = A' \cup B'$.

Example 1. *If $A = \{a, b, c, d\}$, $B = \{b, d, e, f\}$ and $C = \{e, d, g, b\}$, write $A \cup B$, $B \cap C$ and $A - B$.*

Solution: We have,

$$A = \{a, b, c, d\}, B = \{b, d, e, f\}, C = \{e, d, g, b\}$$

$\therefore$ $A \cup B = \{a, b, c, d\} \cup \{b, d, e, f\} = \{a, b, c, d, e, f\}$

$B \cap C = \{b, d, e, f\} \cap \{e, d, g, b\} = \{b, d, e\}$

$A - B = \{a, b, c, d\} - \{b, d, e, f\} = \{a, c\}$

Example 2. *If P is the set of all positive prime numbers and E the set of all positive even numbers; find $P \cap E$.*

Solution: P = set of all + ve prime numbers = $\{2, 3, 5, 7, 11, ...\}$

E = set of all + ve even numbers = $\{2, 4, 6, 8, 10, ...\}$

$\therefore$ $P \cap E = \{2, 3, 5, 7, 11, ...\} \cap \{2, 4, 6, 8, 10, ...\} = \{2\}$

Example 3. *For the following sets, find their union and intersection:*

(i) $A = \{a, e, i, o, u\}$, $B = \{a, b\}$; (ii) $X = \{1, 3, 5\}$, $Y = \{1, 2, 3\}$

(iii) $A = \{x : x \text{ is a natural number and multiple of } 3\}$;
$B = \{x : x \text{ is a natural number less than } 6\}$;

(iv) $A = \{x : x \text{ is a natural number and } 1 < x \leq 6\}$;
$B = \{x : x \text{ is a natural number and } 6 < x < 10\}$;

(v) $A = \{1, 2, 3\}$, $B = \phi$.

Solution: (i) $A \cup B = \{a, e, i, o, u\} \cup \{a, b\} = \{a, e, i, o, u, b\}$

$A \cap B = \{a, e, i, o, u\} \cap \{a, b\} = \{a\}$

(ii) $X \cup Y = \{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 2, 3, 5\}$

$X \cap Y = \{1, 3, 5\} \cup \{1, 2, 3\} = \{1, 3\}$

(iii) $A = \{3, 6, 9, 12, ...\}$, $B = \{1, 2, 3, 4, 5\}$

$A \cup B = \{3, 6, 9, 12, ...\} \cup \{1, 2, 3, 4, 5\} = \{1, 2, 3, 4, 5, 6, 9, 12, ...\}$
$= \{x : x = 1, 2, 4, 5 \text{ or a multiple of } 3\}$

$A \cap B = \{3, 6, 9, 12, ...\} \cap \{1, 2, 3, 4, 5\} = \{3\}$

(iv) $A = \{2, 3, 4, 5, 6\}$, $B = \{7, 8, 9\}$

$A \cup B = \{2, 3, 4, 5, 6\} \cup \{7, 8, 9\} = \{2, 3, 4, 5, 6, 7, 8, 9\}$
$= \{x : 1 < x < 10 \text{ and } x \in N\}$

$A \cap B = \{2, 3, 4, 5, 6\} \cap \{7, 8, 9\} = \phi$

(v) $A \cup B = \{1, 2, 3\} \cup \phi = \{1, 2, 3\} = A$

$A \cap B = \{1, 2, 3\} \cap \phi = \phi$

Example 4. *Given $U = \{x : x \text{ is a natural number}\}$, $B = \{2x : x \in U\}$ and $C = \{2x + 1 : x \in U\}$, find* (i) $B \cup C$; (ii) $B \cap C$; (iii) $U - C$; (iv) B'.

Solution: Evidently, $U = \{1, 2, 3, 4, 5, 6, ...\}$, $B = \{2, 4, 6, 8, ...\}$ and $C = \{3, 5, 7, 9 ...\}$

$\therefore$ (i) $B \cup C = \{2, 3, 4, 5, 6, 7, 8, 9, ...\}$ (ii) $B \cap C = \phi$

(iii) $U - C = \{1, 2, 4, 6, ...\}$ (iv) $B' = \{1, 3, 5, 7, ...\}$.

Example 5. *Let* $A = \{x : x \in N \wedge x \text{ is a multiple of } 2\}$

$B = \{x : x \in N \wedge x \text{ is a multiple of } 5\}$

$C = \{x : x \in N \wedge x \text{ is a multiple of } 10\}$

Describe the sets: (i) $(A \cap B) \cap C$; (ii) $A \cup (B \cap C)$; (iii) $A \cap (B \cup C)$.

Solution: $A = \{x : x \in N \wedge x \text{ is a multiple of } 2\} = \{2, 4, 6, ...\}$

$B = \{x : x \in N \wedge x \text{ is a multiple of } 5\} = \{5, 10, 15, ...\}$

$C = \{x : x \in N \wedge x \text{ is a multiple of } 10\} = \{10, 20, 30, ...\}$

(i) $A \cap B = \{2, 4, 6, ...\} \cap \{5, 10, 15, ...\} = \{10, 20, 30, ...\} = C$

$\therefore \quad (A \cap B) \cap C = C \cap C = C$

(ii) $B \cap C = \{5, 10, 15, ...\} \cap \{10, 20, 30, ...\} = \{10, 20, 30, ...\}$

$\therefore \quad A \cup (B \cap C) = \{2, 4, 6, ...\} \cup \{10, 20, 30, ...\} = \{2, 4, 6, ...\} = A$

(iii) $B \cup C = \{5, 10, 15, ...\} \cup \{10, 20, 30, ...\} = \{5, 10, 15,...\}$

$\therefore \quad A \cap (B \cup C) = \{2, 4, 6, ...\} \cap \{5, 10, 15, ...\} = \{10, 20, 30, ...\} = C$

Example 6. *If* $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, *find the complements of the following sets:*

(i) $A = \{2, 4, 6, 8\}$; (ii) $B = \{1, 3, 5, 7, 9\}$; (iii) $C = \{2, 3, 5, 7\}$;

(iv) ϕ'; (v) U'.

Solution: (i) $A = \{2, 4, 6, 8\}$

A' = set of numbers in U which are not in $A = \{1, 3, 5, 7, 9\}$

(ii) $B = \{1, 3, 5, 7, 9\}$

B' = set of numbers in U which are not in $B = \{2, 4, 6, 8\}$

(iii) $C = \{2, 3, 5, 7\}$

C' = set of numbers in U which are not in $C = \{1, 4, 6, 8, 9\}$

(iv) $\phi = \{\}$, the empty set

ϕ' = set of numbers in U which are not in ϕ

= set of numbers in $U = U$

(v) U' = set of numbers in U which are not in U

= ϕ, since there is no such number.

Example 7. *If* $U = \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, $A = \{2, 4, 7\}$, $B = \{3, 5, 7, 9, 11\}$ *and* $C = \{7, 8, 9, 10, 11\}$, *compute* (i) $(A \cap U) \cap (B \cup C)$; (ii) $C - B$; (iii) $B - C$; (iv) $(B - C)'$.

Solution: (i) Here $A \cap U = \{2, 4, 7\}$; $B \cup C = \{3, 5, 7, 8, 9, 10, 11\}$.

$(A \cap U) \cap (B \cup C) = \{2, 4, 7\} \cap \{3, 5, 7, 8, 9, 10, 11\} = \{7\}$

(ii) $C - B$ is a set of members which belong to C, but do not belong to B.

$\therefore \quad C - B = \{8, 10\}$

(iii) $B - C$ is a set of members which belong to B, but do not belong to C.

$\therefore \quad B - C = \{3, 5\}$

(iv) From (iii),

$B - C = \{3, 5\}$

$\therefore \quad (B - C)' = \{2, 4, 6, 7, 8, 9, 10, 11\}$

Example 8. *Verify* $(A \cap B)' = A' \cup B'$ *where* $A = \{2, 3, 4, 5, 6\}$ *and* $B = \{3, 6, 7, 8\}$ *are subsets of the set* $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

Solution: We have,

$A = \{2, 3, 4, 5, 6\}$, $B = \{3, 6, 7, 8\}$ and $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$\therefore \quad A \cap B = \{3, 6\}$

Then $\quad (A \cap B)' = \{1, 2, 4, 5, 7, 8\} \quad ...(1)$

Also $\quad A' = \{1, 7, 8\}$ and $B' = \{1, 2, 4, 5\}$

$\therefore \quad A' \cup B' = \{1, 2, 4, 5, 7, 8\} \quad ...(2)$

From (1) and (2)

$(A \cap B)' = A' \cup B'$

Example 9. *If* $A = \{2, 4, 6, 8, 10\}$, $B = \{1, 2, 3, 4, 5, 6, 7\}$, $C = \{2, 6, 7, 10\}$ *and* $U = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, *verify that*

(i) $(B \cup C)' = B' \cap C'$; (ii) $A \cup (B \cup C) = (A \cup B) \cup C$;
(iii) $A \cap (B \cap C) = (A \cap B) \cap C$; (iv) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$;
(v) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Solution: (i) $B \cup C = \{1, 2, 3, 4, 5, 6, 7\} \cup \{2, 6, 7, 10\} = \{1, 2, 3, 4, 5, 6, 7, 10\}$

$\therefore$ $(B \cup C)' = \{8, 9\}$

Also $B' = \{8, 9, 10\}$, $C' = \{1, 3, 4, 5, 8, 9\}$

$\therefore$ $B' \cap C' = \{8, 9\}$

$\therefore$ $(B \cup C)' = B' \cap C'$.

(ii) $A = \{2, 4, 6, 8, 10\}$, $B \cup C = \{1, 2, 3, 4, 5, 6, 7, 10\}$

$\therefore$ $A \cup (B \cup C) = \{1, 2, 3, 4, 5, 6, 7, 8, 10\}$

Also $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 10\}$

and $C = \{2, 6, 7, 10\}$

$\therefore$ $(A \cup B) \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 10\}$

$\therefore$ $A \cup (B \cup C) = (A \cup B) \cup C$

(iii) $A = \{2, 4, 6, 8, 10\}$, $B \cap C = \{2, 6, 7\}$

$\therefore$ $A \cap (B \cap C) = \{2, 6\}$

Also $A \cap B = \{2, 4, 6\}$, $C = \{2, 6, 7, 10\}$

$\therefore$ $(A \cap B) \cap C = \{2, 6\}$

$\therefore$ $A \cap (B \cap C) = (A \cap B) \cap C$

(iv) $A = \{2, 4, 6, 8, 10\}$, $B \cap C = \{2, 6, 7\}$

$\therefore$ $A \cup (B \cap C) = \{2, 4, 6, 7, 8, 10\}$

Also $A \cup B = \{1, 2, 3, 4, 5, 6, 7, 8, 10\}$

and $A \cup C = \{2, 4, 6, 7, 8, 10\}$

$\therefore$ $(A \cup B) \cap (A \cup C) = \{2, 4, 6, 7, 8\ 10\}$

$\therefore$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(v) $A = \{2, 4, 6, 8, 10\}$

$B \cup C = \{1, 2, 3, 4, 5, 6, 7, 10\}$

$\therefore$ $A \cap (B \cup C) = \{2, 4, 6, 10\}$

Also $A \cap B = \{2, 4, 6\}$, $A \cap C = \{2, 6, 10\}$

$\therefore$ $(A \cap B) \cup (A \cap C) = \{2, 4, 6, 10\}$

$\therefore$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Example 10. *If* $U = \{a, b, c, d, e, f\}$, $A = \{a, b, c\}$, $B = \{c, d, e, f\}$, $C = \{c, d, e\}$, *and* $D = \{d, e, f\}$, *tabulate the following sets:*

(i) $A \cap D$; (ii) $B \cap D$; (iii) $A \cap C$;
(iv) $U \cap D$; (v) $(U \cap \phi)'$; (vi) $A \cup \phi$;
(vii) $(U \cup A)'$; (viii) $A \cap \phi$; (ix) $A \cap (B \cap C)$;
(x) $(A \cup B) \cup C$; (xi) $A \cup (B \cap C)$; (xii) $(A \cap B) \cup (A \cap C)$.

Solution: (i) $A \cap D = \{a, b, c\} \cap \{d, e, f\} = \phi$

(ii) $B \cap D = \{c, d, e, f\} \cap \{d., e, f\} = \{d, e, f\}$

(iii) $A \cap C = \{a, b, c\} \cap \{c, d, e\} = \{c\}$

(iv) $U \cap D = \{a, b, c, d, e, f\} \cap \{d, e, f\} = \{d, e, f\} = D$

(v) $U \cap \phi = \{a, b, c, d, e, f\} \cap \{\} = \{\} = \phi$

$\therefore$ $(U \cap \phi)'$ = the set of all elements in U which are not in $U \cap \phi$

$= U$ ($\because$ $U \cap \phi$ has no element)

(vi) $A \cup \phi = \{a, b, c\} \cup \{\} = \{a, b, c\} = A$

(vii) $U \cup A = \{a, b, c, d, e, f\} \cup \{a, b, c\} = \{a, b, c, d, e, f\} = U$

$\therefore$ $(U \cup A)' = \phi$

(viii) $A \cap \phi = \{a, b, c\} \cap \{\ \} = \{\ \} = \phi$

(ix) $B \cap C = \{c, d, e, f\} \cap \{c, d, e\} = \{c, d, e\}$

$\therefore$ $A \cap (B \cap C) = \{a, b, c\} \cap \{c, d, e\} = \{c\}$

(x) $A \cup B = \{a, b, c\} \cup \{c, d, e, f\} = \{a, b, c, d, e, f\}$

$\therefore$ $(A \cup B) \cup C = \{a, b, c, d, e, f\} \cup \{c, d, e\} = \{a, b, c, d, e, f\}$

(xi) $B \cap C = \{c, d, e\}$ [See part (ix)]

$\therefore$ $A \cup (B \cap C) = \{a, b, c\} \cup \{c, d, e\} = \{a, b, c, d, e\}$

(xii) $A \cap B = \{a, b, c\} \cap \{c, d, e, f\} = \{c\}$

$A \cap C = \{a, b, c\} \cap \{c, d, e\} = \{c\}$

$\therefore$ $(A \cap B) \cup (A \cap C) = \{c\} \cup \{c\} = \{c\}$

Example 11. *If $A - B = A$, justify $A \cap B = \phi$.*

Solution: Since $A - B = A$,

$\therefore$ A does not contain any element of B. Thus, A and B are disjoint sets (i.e., they have no element common).

$$\therefore \qquad A \cap B = \phi.$$

Example 12. *If $A = \{4, 5, 8, 12\}$, $B = \{1, 4, 6, 9\}$, $C = \{1, 2, 3, 4\}$ then find*
(i) $A - (B - A)$; (ii) $A - (C - B)$

Solution: (i) $B - A = \{1, 4, 6, 9\} - \{4, 5, 8, 12\} = \{1, 6, 9\}$

$\therefore$ $A - (B - A) = \{4, 5, 8, 12\} - \{1, 6, 9\} = \{4, 5, 8, 12\}$.

(ii) $C - B = \{1, 2, 3, 4\} - \{1, 4, 6, 9\} = \{2, 3\}$

$\therefore$ $A - (C - B) = \{4, 5, 8, 12\} - \{2, 3\} = \{4, 5, 8, 12\}$.

Example 13. *Which of the following pairs of sets are disjoint?*

(i) $\{1, 2, 3, 4\}$ *and* $\{x : x$ *is a natural number and* $4 \le x \le 6\}$;

(ii) {a, e, i, o, u} *and* {c, d, e, f};

(iii) $\{x : x$ *is an even integer*$\}$ *and* $\{x : x$ *is an odd integer*$\}$.

Solution: (i) $\{x : x$ is a natural number and $4 \le x \le 6\} = \{4, 5, 6\}$

Now, $\{1, 2, 3, 4\}$ and $\{4, 5, 6\}$ have one element 4 common.

$\therefore$ The given two sets are not disjoint.

(ii) The sets {a, e, i, o, u} and {c, d, e, f} have one element *e* common.

$\therefore$ The given two sets are not disjoint.

(iii) The sets $\{x : x$ is an even integer$\}$

and $\{x : x$ is an odd integer$\}$

have no element common and, therefore, they are disjoint sets.

Example 14. *Show, by an example, $A \cap B = A \cap C$ need not imply $B = C$.*

Solution: Let $A = \{1, 2\}$, $B = \{1, 3\}$ *and* $C = \{1, 4\}$

Now, $A \cap B = \{1, 2\} \cap \{1, 3\} = \{1\}$ and $A \cap C = \{1, 2\} \cap \{1, 4\} = \{1\}$

$\therefore \quad A \cap B = A \cap C$, but $B \neq C$.

Example 15. *Let A, B and C be the sets such that $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Show that $B = C$.*

Solution: We have,

$$A \cup B = A \cup C \Rightarrow (A \cup B) \cap C = (A \cup C) \cap C$$
$$\Rightarrow (A \cap C) \cup (B \cap C) = C \qquad [\because (A \cup C) \cap C = C]$$
$$\Rightarrow (A \cap B) \cup (B \cap C) = C \qquad ...(1)$$
$$[\because A \cap C = A \cap B]$$

Again,

$$A \cup B = A \cup C \Rightarrow (A \cup B) \cap B = (A \cup C) \cap B$$
$$\Rightarrow B = (A \cap B) \cup (C \cap B) \qquad [\because (A \cup B) \cap B = B]$$
$$\Rightarrow B = (A \cap B) \cup (B \cap C) \qquad ...(2)$$

From (1) and (2), we get $B = C$.

Example 16. *For any two sets A and B, prove that $P(A) = P(B) \Rightarrow A = B$.*

Solution: Let x be an arbitrary element of A. Then, there exists a subset, say X, of set A such that $x \in X$. Now,

$$X \subset A \Rightarrow X \in P(A)$$
$$\Rightarrow X \in P(B) \qquad [\because P(A) = P(B)]$$
$$\Rightarrow X \subset B$$
$$\Rightarrow x \in B \qquad [\because x \in X \text{ and } X \subset B \therefore x \in B]$$

Thus, $\quad x \in A \Rightarrow x \in B$

$\therefore \quad A \subseteq B \qquad ...(1)$

Now, let y be an arbitrary element of B. Then, there exists a subset, say Y, of set B such that $y \in Y$.

Now,

$$Y \subset B \Rightarrow Y \in P(B)$$
$$\Rightarrow Y \in P(A) \qquad [\because P(A) = P(B)]$$
$$\Rightarrow Y \subset A$$
$$\Rightarrow y \in A$$

Thus, $\quad y \in B \Rightarrow y \in A$

$\therefore \quad B \subseteq A \qquad ...(2)$

From (1) and (2), we obtain $A = B$.

Example 17. *Let A and B be sets, if $A \cap X = B \cap X = \phi$ and $A \cup X = B \cup X$ for some set X, prove that $A = B$.*

Solution: We have for some set X,

$$A \cup X = B \cup X \Rightarrow A \cap (A \cup X) = A \cap (B \cup X)$$
$$\Rightarrow A = (A \cap B) \cup (A \cap X) \qquad [\because A \cap (A \cup X) = A]$$
$$\Rightarrow A = (A \cap B) \cup \phi \qquad [\because A \cap X = \phi \text{ (given)}]$$
$$\Rightarrow A = A \cap B$$
$$\Rightarrow A \subseteq B \qquad ...(1)$$

Again,

$$A \cup X = B \cup X \Rightarrow B \cap (A \cup X) = B \cap (B \cup X)$$
$$\Rightarrow (B \cap A) \cup (B \cap X) = B \qquad [\because B \cap (B \cup X) = B]$$
$$\Rightarrow (B \cap A) \cup \phi = B \qquad [\because B \cap X = \phi \text{ (given)}]$$
$$\Rightarrow B \cap A = B$$
$$\Rightarrow A \cap B = B$$
$$\Rightarrow B \subseteq A \qquad ...(2)$$

From (1) and (2), we get $A = B$.

Example 18. *Prove:*

(i) $A \cup (A \cap B) = A$; (ii) $A \cap (A \cup B) = A$; (iii) $(A - B) \cap (B - A) = \phi$.

Solution: (i) Let x be any arbitrary element of $A \cup (A \cap B)$.

Then $x \in A \cup (A \cap B) = \{x : x \in A \text{ or } x \in (A \cap B)\}$
$= \{x : x \in A \text{ or } (x \in A, x \in B)\}$
$= \{x : x \in A\}$
$= A$ Hence proved.

(ii) Let x be any arbitrary element of $A \cap (A \cup B)$.

Then $x \in A \cap (A \cup B)$
$= \{x : x \text{ is the element common to } A \text{ and } A \cup B\}$
$= \{x : x \in A\}$ $[\because A \cup B = \{x : x \in A \text{ or } x \in B\}]$
$= A$

(iii) Any $x \in A - B \Rightarrow x \in A$ and $x \notin B \Rightarrow x \notin B - A$

Further any $x \in B - A \Rightarrow x \in B$ and $x \notin A \Rightarrow x \notin A - B$

This proves that $(A - B) \cap (B - A) = \phi$.

Example 19. *If A, B, C are three sets such that $A \subset B$, then prove that $C - B \subset C - A$.*

Solution: Let $x \in C - B$. Then

$$x \in C - B \Rightarrow x \in C \text{ and } x \notin B$$
$$\Rightarrow x \in C \text{ and } x \notin A \qquad [\because A \subset B]$$
$$\Rightarrow x \in C - A \qquad \therefore \quad C - B \subset C - A$$

Example 20. *For any two sets A and B, prove that $A \cup B = A \cap B$ if and only if $A = B$.*

Solution: Let $A = B$.

Then $A \cup B = A$ and $A \cap B = A$
$\Rightarrow A \cup B = A \cap B$

Thus, $A = B \Rightarrow A \cup B = A \cap B$ ... (1)

Conversely, let $A \cup B = A \cap B$.

To prove: $A = B$.

Let $x \in A \Rightarrow x \in A \cup B$
$\Rightarrow x \in A \cap B$ $[\because A \cup B = A \cap B]$
$\Rightarrow x \in A$ and $x \in B$
$\Rightarrow x \in B$

$\therefore A \subset B$ ... (2)

Now, let $y \in B \Rightarrow y \in A \cup B$
$\Rightarrow y \in A \cap B$ $[\because A \cup B = A \cap B]$
$\Rightarrow y \in A$ and $y \in B$
$\Rightarrow y \in A$

$\therefore B \subset A$ ... (3)

From (2) and (3), we get $A = B$.

Thus, $A \cup B = A \cap B \Rightarrow A = B$ ... (4)

From (1) and (4), we have

$A \cup B = A \cap B \Leftrightarrow A = B$

Example 21. *Show that for any sets A and B, $A = (A \cap B) \cup (A - B)$ and $A \cup (B - A) = A \cup B$.*

Solution: $(A \cap B) \cup (A - B) = (A \cap B) \cup (A \cap B')$ $[\because A - B = A \cap B']$

$= A \cap (B \cup B')$ [Distributive law]

$= A \cap X$, where X is universal set $= A$

$A \cup (B - A) = A \cup (B \cap A')$ $[\because B - A = B \cap A']$

$= (A \cup B) \cap (A \cup A')$ [Distributive law]

$= (A \cup B) \cap X$, where $X = A \cup A'$ is universal set

$= A \cup B$ $(\because A \cup B \subset X)$

Example 22. *Draw appropriate Venn diagram for each of the following:*

(i) $(A \cup B)'$; (ii) $A' \cap B'$; (iii) $(A \cap B)'$; (iv) $A' \cup B'$.

Solution: (i) Shaded area $(A \cup B)'$

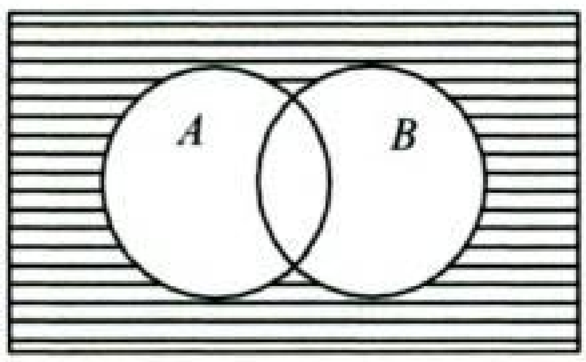

Fig. 1.18

(ii) Shaded are shown in Fig. 1.18 above.

(iii) Shaded are $(A \cap B)'$

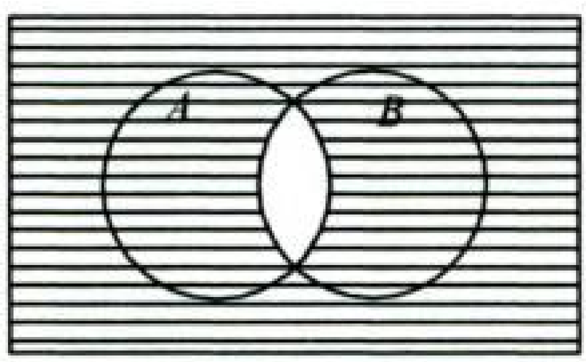

Fig. 1.19

(iv) Shaded are shown in Fig. 1.19 above.

Example 23 *Show that $A \cap B = A \cap C$ need not imply $B = C$.*

Solution: With the help of an example, we may try to establish it. Let $A = \{1, 2\}$, $B = \{1, 3\}$ and $C = \{1, 4\}$. Now,

$$A \cap B = \{1, 2\} \cap \{1, 3\} = \{1\}$$

and $$A \cap C = \{1, 2\} \cap \{1, 4\} = \{1\}$$

$\therefore$ $$A \cap B = A \cap C$$

still $$B \neq C$$

Example 24 *Find set A, B and C such that $A \cap B$, $A \cap C$ and $B \cap C$ are non-empty sets and $A \cap B \cap C = \phi$.*

Solution: Let $A = \{1, 2\}$, $B = \{2, 3\}$ and $C = \{1, 3\}$. Clearly $A \cap B = \{2\}$, $B \cap C = \{3\}$ and $A \cap C = \{1\}$

i.e., $A \cap B$, $A \cap C$ and $B \cap C$ are non-empty sets.

$\therefore \qquad (A \cap B) \cap C = \{2\} \cap \{1, 3\} \Rightarrow A \cap B \cap C = \phi.$

EXERCISE 1.4

LEVEL OF DIFFICULTY A

1. (i) If $A = \{1, 2, 3, 4\}$; $B = \{3, 4, 5\}$; $C = \{4, 5, 7, 8\}$;
find (a) $A \cap B$; (b) $B \cap C$; (c) $B - C$.

(ii) Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$; $A = \{1, 2, 3, 4\}$; $B = \{2, 4, 6, 8\}$; $C = \{3, 4, 5, 6\}$.
Find (a) $(A \cap C)'$; (b) $(B \cap C)'$; (c) $(A - C)'$.

(iii) If $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$; $A = \{1, 2, 3, 4\}$; $B = \{3, 4, 6\}$;
$C = \{5, 6, 7, 8\}$, verify the following:

(a) $A \cup (B \cup C) = (A \cup B) \cup C$; (b) $A \cap (B \cap C) = (A \cap B) \cap C$;
(c) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; (d) $(A \cap B)' = A' \cup B'$;
(e) $(A \cup B)' = A' \cap B'$.

(iv) If $A = \{a, b, c, d, e\}$ and $B = \{d, e, f, g\}$, find $(A - B) \cap (B - A)$.

2. If $A = \{2, 4, 6, 8, 10\}$, $B = \{1, 2, 3, 4, 5, 6, 7\}$ and $C = \{2, 6, 7, 10\}$, verify that

(i) $A - (B \cup C) = (A - B) \cap (A - C)$; (ii) $A - (B \cap C) = (A - B) \cup (A - C)$.

3. If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{1, 3, 4\}$ and $B = \{5, 6\}$, verify that $A - B = A \cap B' = B' - A'$.

4. (a) Find the union of each of the following pairs of sets:

(i) $A = \{a, e, i, o, u\}$, $B = \{a, b, c\}$. (ii) $A = \{1, 3, 5\}$, $B = \{1, 2, 3\}$;
(iii) $A = \{x : x$ is a natural number and multiple of $3\}$,
$B = \{x : x$ is a natural number less than $6\}$;
(iv) $A = \{x : x$ is a natural number and $1 < x \leq 6\}$,
$B = \{x : x$ is a natural number and $6 < x \leq 10\}$;
(v) $A = \{1, 2, 3\}$, $B = \phi$.

(b) Find also the intersection of each pairs of sets given above.

5. If $A = \{3, 5, 7, 9, 11\}$, $B = \{7, 9, 11, 13\}$, $C = \{11, 13, 15\}$, and $D = \{15, 17\}$, find

(i) $A \cap B$; (ii) $B \cap C$; (iii) $A \cap C \cap D$;
(iv) $A \cap C$; (v) $B \cap D$; (vi) $A \cap (B \cap C)$;
(vii) $A \cap D$; (viii) $A \cap (B \cup D)$; (ix) $(A \cap B) \cap (B \cup C)$;
(x) $(A \cup D) \cap (B \cup C)$.

6. If $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{5, 6, 7, 8\}$ and $D = \{7, 8, 9, 10\}$, find

(i) $A \cup B$; (ii) $A \cup C$; (iii) $B \cup C$; (iv) $B \cup D$;
(v) $A \cup B \cup C$; (vi) $A \cup B \cup D$; (vii) $B \cup C \cup D$.

7. Which of the following pairs of sets are disjoint?

(i) $\{1, 2, 3, 4\}$ and $\{x : x$ is a natural number and $4 \leq x \leq 6\}$;
(ii) {a, e, i, o, u} and $\{c, d, e, f\}$;
(iii) $\{x : x$ is an even integer$\}$ and $\{x : x$ is an odd integer$\}$.

8. State whether each of the following statements are true of false. Justify your answer.

(i) $\{2, 3, 4, 5\}$ and $\{3, 6\}$ are disjoint sets.
(ii) {a, e, i, o, u} and $\{a, b, c, d\}$ are disjoint sets.

(iii) $\{2, 6, 10, 14\}$ and $\{3, 7, 11, 15\}$ are disjoint sets.
(iv) $\{2, 6, 10\}$ and $\{3, 7, 11\}$ are disjoint sets.

9. Taking the set of natural numbers as the universal set, write down the complements of the following sets:
(i) $\{x : x \in N$ and x is even$\}$; (ii) $\{x : x \in N$ and x is odd$\}$;
(iii) $\{x : x \in N$ and $x = 3n$ for some $n \in N\}$;
(iv) $\{x : x$ is a prime number$\}$; (v) $\{x : x \in N$ and x is a perfect square$\}$;
(vi) $\{x : x \in N$ and x is a perfect cube$\}$; (vii) $\{x : x \in N$ and $x + 5 = 8\}$;
(viii) $\{x : x \in N$ and $2x + 5 = 9\}$; (ix) $\{x : x \in N$ and $x \geq 7\}$;
(x) $\{x : x \in N$ and x is a divisible by 3 and 5$\}$; (xi) $\{x : x \in N$ and $2x + 1 > 10\}$.

10. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{2, 4, 6, 8\}$ and $B = \{2, 3, 5, 7\}$. Verify that
(i) $(A \cup B)' = A' \cap B'$; (ii) $(A \cap B)' = A' \cup B'$.

11. Let $A = \{3, 6, 12, 15, 18, 21\}$, $B = \{4, 8, 12, 16, 20\}$, $C = \{2, 4, 6, 8, 10, 12, 14, 16\}$, and $D = \{5, 10, 15, 20\}$. Find
(i) $A - B$; (ii) $A - C$; (iii) $A - D$; (iv) $B - A$;
(v) $C - A$; (vi) $D - A$; (vii) $B - C$; (viii) $B - D$;
(ix) $C - B$; (x) $D - B$; (xi) $C - D$; (xii) $D - C$.

12. Let $A = \{x : x$ is a natural number$\}$, $B = \{x : x$ is an even natural number$\}$, $C = \{x : x$ is an odd natural number$\}$ and $D = \{x : x$ is a prime number$\}$. Find
(i) $A \cap B$; (ii) $A \cap C$; (iii) $A \cap D$; (iv) $B \cap C$;
(v) $B \cap D$; (vi) $C \cap D$.

13. If $X = \{a, b, c, d\}$ and $Y = \{f, b, d, g\}$, find,
(i) $X - Y$; (ii) $Y - X$; (iii) $X \cap Y$.

14. If R is the set of real numbers and Q is the set of rational numbers, then what is $R - Q$?

15. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{1, 2, 3, 4,\}$, $B = \{2, 4, 6, 8\}$ and $C = \{3, 4, 5, 6\}$. Find (i) A'; (ii) B'; (iii) $(A \cap C)'$; (iv) $(A \cup B)'$; (v) $(A')'$; (vi) $(B - C)'$.

16. (a) If $U = \{a, b, c, d, e, f, g, h\}$, find the complements of the following sets:
(i) $A = \{a, b, c\}$; (ii) $B = \{d, e, f, g\}$; (iii) $C = \{a, c, e, g\}$; (iv) $D = \{f, g, h, a\}$.

(b) Let U be the set of all triangles in a plane. If A is the set of all triangles with atleast one angle different from 60°, what is A'?

17. Let $A = \{a, b\}$ and $B = \{a, b, c\}$. Is $A \subset B$? What is $A \cup B$?

18. Shade the following sets in the Venn diagram:
(i) $A' \cap (B \cup C)$; (ii) $A' \cap (C - B)$.

19. Let $A = \{1, 2, 3, 4\}$, $B = \{4, 6, 7, 8\}$ and $C = \{2, 4, 6, 8\}$. Verify the following identities:
(i) $(A \cup B) \cup C = A \cup (B \cup C)$; (ii) $(A \cap B) \cap C = A \cap (B \cap C)$;
(iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$; (iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

20. Let $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 8\}$, $C = \{3, 4, 5, 6\}$ and the universal set $U = \{1, 2, 3, 4, ..., 9\}$. Verify that
(i) $A \cap (B \cap C) = (A \cap B) \cap C$; (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$;
(iii) $(A \cup B)' = A' \cap B'$; (iv) $A' \cup B = (A \cap B')'$.

21. If A and B are two sets such that $A \subset B$, then what is $A \cup B$?

LEVEL OF DIFFICULTY B

22. Let A, B and C be three sets such that $A \cup B = C$ and $A \cap B = \phi$. Then, prove that $A = C - B$.

23. For any two sets A and B prove by using properties of sets that:
(i) $(A \cup B) - (A \cap B) = (A - B) \cup (B - A)$; (ii) $(A \cap B) \cup (A - B) = A$;
(iii) $(A \cup B) - A = B - A$.

24. Let A and B be any two sets. Using properties of sets prove that:
(i) $(A - B) \cup B = A \cup B$; (ii) $(A - B) \cup A = A$;
(iii) $(A - B) \cap B = \phi$; (iv) $(A - B) \cap A = A \cap B'$.

25. For sets A, B and C using properties of sets, prove that:
(i) $(A \cup B) - C = (A - C) \cup (B - C)$; (ii) $(A \cup B) - A = B - A$;
(iii) $A - (B - C) = (A - B) \cup (A \cap C)$; (iv) $A \cap (B - C) = (A \cap B) - (A \cap C)$.

26. Show that the following four conditions are equivalent:
(i) $A \subset B$; (ii) $A - B = \phi$; (iii) $A \cup B = B$; (iv) $A \cap B = A$.

27. Is it true that, for any sets A and B, $P(A) \cup P(B) = P(A \cup B)$? Justify your answer.

28. If A and B are respectively the sets having the elements as the zeros of the polynomials $x^3 - 4x^2 + x + 6$ and $x^3 - 6x^2 + 11x - 6$, find
(i) $A - B$; (ii) $B - A$; (iii) $A - (B - A)$.

29. For any two sets A and B, prove that $P(A) \cup P(B) \subset P(A \cup B)$, but, $P(A \cup B)$ is not necessarily a subset of $P(A) \cup P(B)$.

30. For any two sets A and B, prove that $P(A \cap B) = P(A) \cap P(B)$.

31. Two finite sets have m and n elements. The total number of subsets of the first set is 56 more than the total number of subsets of the second set. Find the values of m and n.

32. If $aN = \{ax : x \in N\}$, then find $3N \cap 7N$.

33. If $X = \{4^n - 3n - 1 : n \in N\}$ and $Y = \{9(n - 1)\}: n \in N$, prove that $X \subset Y$.

34. For any natural number a, we define $aN = \{ax : x \in N\}$. If $b, c, d \in N$ such that $bN \cap cN = dN$, prove that d is the l.c.m. of b and c.

35. Suppose $A_1, A_2, ..., A_{30}$ are thirty sets each with five elements and $B_1, B_2, ..., B_n$ are n sets each with three elements. Let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^{n} B_j = S$. Assume that each element of S belongs to exactly ten of the A_i's and exactly 9 of B_j's. Find n.

36. Using properties of sets, show that for any two sets A and B,
$(A' \cup B) \cap (A \cup B') = A$

37. If $A' \cup B = U$, show that $A \subset B$.

38. If $B' \subseteq A'$, show that $A \subseteq B$.

39. For any two sets A and B, prove that
(i) $(A - B) \cup B = A \cup B$ (ii) $(A - B) \cap B = \phi$
(iii) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

40. Prove that $A - (B \cup C) = (A - B) \cap (A - C)$.

41. Prove that $A' - B' = B - A$.

Answers

1. (i) (a) {3, 4}, (b) {4, 5}, (c) {3};
(ii) (a) {1, 2, 5, 6, 7, 8, 9}, (b) {1, 2, 3, 5, 7, 8, 9}, (c) {3, 4, 5, 6, 7, 8, 9};
(iv) ϕ.

4. (a)(i){a, e, i, o, u, b, c}; (ii) {1, 2, 3, 5}; (iii) {1, 2, 4, 5, 3, 6, 9, 12, ...};
(iv) {2, 3, 4, 5, 6, 7, 8, 9, 10}; (v) {1, 2, 3}.
(b) (i) {a}; (ii) {1, 3}; (iii) {3}; (iv) ϕ; (v) ϕ.

5. (i) {7, 9, 11}; (ii) {11, 13}; (iii) ϕ; (iv) {11};
(v) ϕ; (vi) {11}; (vii) ϕ; (viii) {7, 9, 11};
(ix) {7, 9, 11}; (x) {7, 9, 11, 15}.

6. (i) {1, 2, 3, 4, 5, 6}; (ii) {1, 2, 3, 4, 5, 6, 7, 8};
(iii) {3, 4, 5, 6, 7, 8}; (iv) {3, 4, 5, 6, 7, 8, 9, 10};
(v) {1, 2, 3, 4, 5, 6, 7, 8}; (vi) {1, 2, 3, 4, 5, 6, 7, 8, 9, 10};
(vii) {3, 4, 5, 6, 7, 8, 9, 10}.

7. (iii). **8.** (i) False; (ii) False; (iii) True; (iv) True.

9. (i) {x : x is an odd natural number}; (ii) {x : x is an even natural number};
(iii) {x : $x \in N$ and x is not a multiple of 3};
(iv) {x : x is a positive composite number and $x = 1$};
(v) {x : $x \in N$ and x is not a perfect square};
(vi) {x : $x \in N$ and x is not a perfect cube};
(vii) {x : $x \in N$ and $x \neq 3$}; (viii) {x : $x \in N$ and $x \neq 2$};
(ix) {x : $x \in N$ and $x < 7$}; (x) {x : $x \in N$ and x is neither divisible by 3 nor by 5);
(xi) {x : $x \in N$ and $x < 9/2$}.

11. (i) {3, 6, 15, 18, 21}; (ii) {3, 15, 18, 21}; (iii) {3, 6, 12, 18, 21};
(iv) {4, 8, 16, 20}; (v) {2, 4, 8, 10, 14, 16}; (vi) {5, 10, 20};
(vii) {20}; (viii) {4, 8, 12, 16}; (ix) {2, 6, 10, 14};
(x) {5, 10, 15}; (xi) {2, 4, 6, 8, 12, 14, 16}; (xii) {5, 15, 20}.

12. (i) B; (ii) C; (iii) D; (iv) ϕ; (v) {2}; (vi) {x : x is an odd prime number}.

13. (i) {a, c}; (ii) {f, g}; (iii) {b, d}.

14. Set of irrational numbers.

15. (i) {5, 6, 7, 8, 9}; (ii) {1, 3, 5, 7, 9}; (iii) {1, 2, 5, 6, 7, 8, 9};
(iv) {5, 7, 9}; (v) {1, 2, 3, 4}; (vi) {1, 3, 4, 5, 6, 7, 9}.

16.(a) (i){d, e, f, g, h}}; (ii) {a, b, c, h}; (iii) {b, d, f, h}; (iv) {b, c, d, e}.
(b) Set of all equilateral triangles.

17. Yes; $A \cup B$ = {a, b, c}.

18. (i)

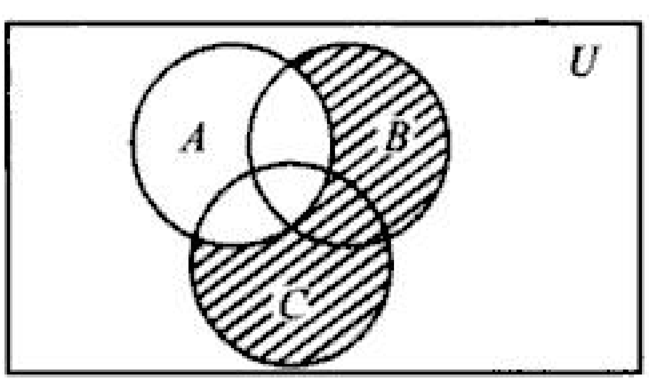

Fig. 1.20

(ii)

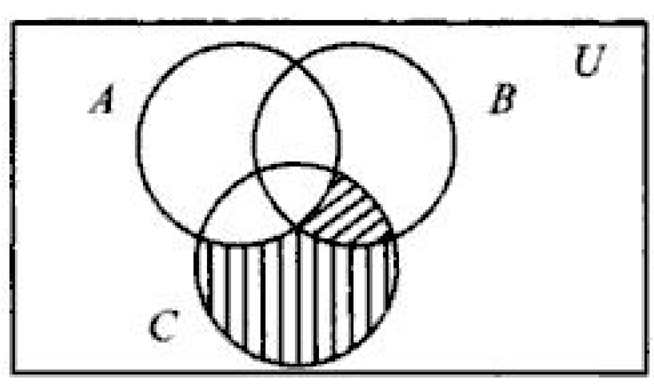

Fig. 1.21

21. B. **27.** No.

31. (i) {−1}; (ii) {1}; (iii) {−1, 2, 3}. **34.** $n = 3$, $m = 6$. **37.** 21 N.

HINTS AND SOLUTIONS

22. We have,

$$A \cup B = C$$

$$\therefore \quad C - B = (A \cup B) - B$$
$$= (A \cup B) \cap B' \quad [\because X - Y = X \cap Y']$$
$$= (A \cap B') \cup (B \cap B')$$
$$= (A \cap B') \cup \phi$$
$$= A \cap B'$$
$$= A - B$$
$$= A \quad [\because A \cap B = \phi]$$

23. (i) We have,

$$(A \cup B) - (A \cap B) = (A \cup B) \cap (A \cap B)' \quad [\because X - Y = X \cap Y']$$
$$= (A \cup B) \cap (A' \cup B') \quad [\because (A \cap B)' = A' \cup B']$$
$$= X \cap (A' \cup B'), \text{ where } X = A \cup B = (X \cap A') \cup (X \cap B')$$
$$= (B \cap A') \cup (A \cap B') \quad \left[\begin{array}{l}\because X \cap A' = (A \cup B) \cap A' \\ = (A \cap A') \cup (B \cap A') = \phi \cup (B \cap A') \\ = B \cap A' \text{ Similarly, } X \cap B' = A \cap B'\end{array}\right]$$
$$= (A \cap B') \cup (B \cap A')$$
$$= (A - B) \cup (B - A) \quad [\because A - B = A \cap B' \text{ and } B - A = B \cap A']$$

(ii) $(A \cap B) \cup (A - B) = (A \cap B) \cup (A \cap B') = X \cup (A \cap B')$, where $X = A \cap B$

$$= (X \cup A) \cap (X \cup B') = A \cap (A \cup B')$$
$$\left[\begin{array}{l}\because X \cup A = (A \cap B) \cup A = A\ [\because A \cap B \subset A] \\ X \cup B' = (A \cap B) \cup B' = (A \cup B') \cap (B \cup B') \\ = (A \cup B') \cap U = A \cup B'\end{array}\right]$$
$$= A \quad [\because A \subset A \cup B']$$

(iii) $(A \cup B) - A = (A \cup B) \cap A' \quad [\because X - Y = X \cap Y']$

$$= (A \cap A') \cup (B \cap A') = \phi \cup (B \cap A') \quad [\because A \cap A' = \phi]$$
$$= B \cap A' = B - A \quad [\because B - A = B \cap A']$$

24. (i) $(A - B) \cup B = (A \cap B') \cup B \quad [\because A - B = A \cap B']$

$$= (A \cup B) \cap (B' \cup B) \quad [\because \cup \text{ is distributive over } \cap]$$
$$= (A \cup B) \cap U \quad [\because B' \cup B = U]$$
$$= A \cup B$$

(ii) $(A - B) \cup A = A \quad [\because A - B \subset A]$

(iii) $(A - B) \cap B = (A \cap B') \cap B = A \cap (B' \cap B) = A \cap \phi = \phi$

(iv) $(A - B) \cap A = A - B \quad [\because A - B \subset A]$

$$= A \cap B'$$

25. (i) $(A \cup B) - C = (A \cup B) \cap C' \quad [\because A - B = A \cap B']$

$$= (A \cap C') \cup (B \cap C')$$
$$= (A - C) \cup (B - C)$$

(ii) $(A \cup B) - A = (A \cup B) \cap A' \quad [\because A - B = A \cap B']$

$$= (A \cap A') \cup (B \cap A')$$
$$= \phi \cup (B \cap A')$$
$$= B \cap A'$$
$$= B - A$$

(iii) $A - (B - C) = A - (B \cap C')$ $[\because B - C = B \cap C']$

$= A \cap (B \cap C')'$

$= A \cap (B' \cup (C')')$ [De-Morgan's Law]

$= A \cap (B' \cup C)$ $[\because (C')' = C]$

$= (A \cap B') \cup (A \cap C)$

$= (A - B) \cup (A \cap C)$

(iv) Let $x \in A \cap (B - C)$ be any element. Then

$x \in A \cap (B - C) \Rightarrow x \in A$ and $x \in (B - C)$

$\Rightarrow x \in A$ and ($x \in B$ and $x \notin C$)

$\Rightarrow$ ($x \in A$ and $x \in B$) and ($x \in A$ and $x \notin C$)

$\Rightarrow x \in (A \cap B)$ and $x \notin (A \cap C)$

$\Rightarrow x \in (A \cap B) - (A \cap C)$

$\therefore \quad A \cap (B - C) \subseteq (A \cap B) - (A \cap C)$ ...(1)

Let $y \in (A \cap B) - (A \cap C)$ be any element. Then

$y \in (A \cap B) - (A \cap C) \Rightarrow y \in A \cap B$ and $y \notin A \cap C$

$\Rightarrow$ ($y \in A$ and $y \in B$) and $y \notin (A \cap C)$

$\Rightarrow$ ($y \in A$ and $y \in B$) and ($y \in A$ and $y \notin C$)

$\Rightarrow y \in A$ and ($y \in B$ and $y \notin C$)

$\Rightarrow y \in A$ and $y \in (B - C)$

$\Rightarrow y \in A \cap (B - C)$

$\therefore \quad (A \cap B) - (A \cap C) \subseteq A \cap (B - C)$ ...(2)

From (1) and (2), we have

$$A \cap (B - C) = (A \cap B) - (A \cap C)$$

26. (i) $\Leftrightarrow$ (ii). That is, $A \subset B \Leftrightarrow$ All elements of A are in $B \Leftrightarrow A - B = \phi$.

(ii) $\Leftrightarrow$ (iii). That is, $A - B = \phi \Leftrightarrow$ All elements of A are in $B \Leftrightarrow A \cup B = B$.

(iii) $\Leftrightarrow$ (iv). That is,

$A \cup B = B \Leftrightarrow$ All the elements of A are in B

$\Leftrightarrow$ All the elements of A are common in A and B

$\Leftrightarrow A \cap B = A$

Thus, all the four given conditions are equivalent.

27. Let $A = \{a\}$, $B = \{b\}$ and $A \cup B = \{a, b\}$

$\therefore \quad P(A) = \{\phi, \{a\}\}; \ P(B) = \{\phi, \{b\}\}$

and $\quad P(A \cup B) = \{\phi, \{a\}, \{b\}, \{a, b\}\}$ ...(1)

and $\quad P(A) \cup P(B) = \{\phi, \{a\}, \{b\}\}$ ...(2)

From (1) and (2), we have

$P(A \cup B) \neq P(A) \cup P(B)$

28. $A = \{-1, 2, 3\}$ and $B = \{1, 2, 3\}$.

29. Let $X \in P(A) \cup P(B)$. Then, $X \in P(A) \cup P(B)$.

$X \in P(A) \cup P(B) \Rightarrow X \in P(A)$ or $X \in P(B)$

$\Rightarrow X \subset A$ or $X \subset B$

$\Rightarrow X \subset A \cup B$

$\Rightarrow X \in P(A \cup B)$

$\therefore \quad P(A) \cup P(B) \subset P(A \cup B)$

Let $A = \{1, 2\}$ and $B = \{3, 4, 5\}$. Then, we find that $X = \{1, 2, 3, 4\} \subset (A \cup B)$. Therefore, $X \in P(A \cup B)$. But, $X \notin P(A)$, $X \notin P(B)$. So, $X \notin P(A) \cup P(B)$.

Thus, $P(A \cup B)$ is not necessarily a subset of $P(A) \cup P(B)$.

30. Let $X \in P(A \cap B) \Rightarrow X \subset A \cap B$

$\Rightarrow x \subset A$ and $X \subset B$

$\Rightarrow X \in P(A)$ and $X \in P(B)$

$\Rightarrow X \in P(A) \cap P(B)$

$\therefore \quad P(A \cap B) \subset P(A) \cap P(B)$ ...(1)

Now, let $\quad Y \in P(A) \cap P(B)$

Then $\quad Y \in P(A) \cap P(B) \Rightarrow Y \in P(A)$ and $Y \in P(B)$

$\Rightarrow Y \subset A$ and $Y \subset B$

$\Rightarrow Y \subset A \cap B$

$\Rightarrow Y \in P(A \cap B)$

$\therefore \quad P(A) \cap P(B) \subset P(A \cap B)$ ...(2)

From (1) and (2), we get

$$P(A \cap B) = P(A) \cap P(B)$$

31. Let A and B be two sets having m and n elements respectively. Then, number of subsets of $A = 2^m$, and number of subsets of $B = 2^n$.

It is given that, $2^m - 2^n = 56 \Rightarrow 2^n(2^{m-n} - 1) = 2^3(2^3 - 1)$

$\Rightarrow n = 3$ and $m - n = 3$

$\Rightarrow n = 3$ and $m = 6$

32. We have $3N = \{3x : x \in N\} = \{3, 6, 9, 12, ...\}$

and $\quad 7N = \{7x : x \in N\} = \{7, 14, 21, 28, 35, 42...\}$

Hence $\quad 3N \cap 7N = \{21, 42, 63, ...\} = \{21x : x \in N\} = 21N$

33. For $n = 1$, $4^n - 3n - 1 = 4 - 3 - 1 = 0$.

For $n \geq 2$, we have

$4^n - 3n - 1 = (1 + 3)^n - 3n - 1$

$= {}^nC_0 + {}^nC1 \cdot 3 + {}^nC_2 \cdot 3^2 + {}^nC_3 \cdot 3^3 + ... + {}^nC_n \cdot 3^n - 3n - 1$

[Using Binomial Theorem]

$= 1 + 3n + {}^nC_2 \cdot 3^2 + {}^nC_3 \cdot 3^3 + ... + {}^nC_n \cdot 3^n - 3n - 1$

$= 3^2[{}^nC_2 + {}^nC_3 \cdot 3 + {}^nC_4 \cdot 3^2 + ... + {}^nC_n \cdot 3^{n-2}]$

$= 9[{}^nC_2 + {}^nC_3 \cdot 3 + {}^nC_4 \cdot 3^2 + ... + {}^nC_n \cdot 3^{n-2}]$

Thus, X consists of all those positive integral multiples of 9 which are of the form $9\,[{}^nC_2 + 3 \cdot {}^nC_3 + 3^2 \cdot {}^nC_4 + ... + 3^{n-2} \cdot {}^nC_n]$ together with 0.

And, y consists of all integral multiples of 9 together with 0.

Thus, $X \subset Y$

34. $bN = \{bx : x \in N\}$ = the set of positive integral multiples of b

$cN = \{cx : x \in N\}$ = the set of positive integral multiples of c

$\therefore bN \cap cN$ = the set of positive integral multiples of b and c both

$= \{dx : x \in N\}$, where d = l.c.m. of b and c

$= dN$

35. Since each A_i has 5 elements and each element of S belongs to exactly 10 of A_i's.

$$\therefore \; S = \bigcup_{i=1}^{30} A_i \Rightarrow n(s) = \frac{1}{10}\sum_{i=1}^{30} n(A_i) = \frac{1}{10}(5\times 30) = 15 \qquad \text{... (1)}$$

Again, each B_j has 3 elements and each element of S belongs to exactly 9 of B_j's.

$$\therefore \; S = \bigcup_{j=1}^{n} B_j \Rightarrow n(s) = \frac{1}{9}\sum_{j=1}^{n} n(B_j) = \frac{1}{9}(3n) = \frac{n}{3} \qquad \text{... (2)}$$

From (1) and (2), we get

$$15 = \frac{n}{3} \Rightarrow n = 45$$

36. $(A \cup B) \cap (A \cup B') = ((A \cup B) \cap A) \cup ((A \cup B) \cap B')$

$= A \cup ((A \cup B') \cap B')$

$= A \cup ((A \cap B') \cup (B \cap B'))$

$= A \cup (A \cap B') = A$

40. (i) Let $x \in (A - B) \cup B \Rightarrow x \in (A - B)$ or $x \in B$

$\Rightarrow (x \in A$ and $x \notin B)$ or $x \in B$

$\Rightarrow (x \in A$ or $x \in B)$ and $(x \notin B$ or $x \in B)$

$\Rightarrow x \in (A \cup B)$

$\therefore \quad (A - B) \cup B \subseteq A \cup B$ ...(1)

Again, let $y \in A \cup B$. Then

$y \in A \cup B \Rightarrow y \in A$ or $y \in B$

$\Rightarrow (y \in A$ or $y \in B)$ and $(y \notin B$ or $y \in B)$

$\Rightarrow (y \in A$ and $y \notin B)$ or $y \in B$

$\Rightarrow y \in (A - B)$ or $y \in B$

$\Rightarrow y \in (A - B) \cup B$

$\therefore \quad A \cup B \subseteq (A - B) \cup B$ ...(2)

From (1) and (2), we get $(A - B) \cup B = A \cup B$

(ii) There are only two possibilities: either (i) $(A - B) \cap B \neq \phi$ or (ii) $(A - B) \cap B = \phi$. Let us take $(A - B) \cap B \neq \phi$. Then

$(A - B) \cap B \neq \phi \Rightarrow$ there exists $x \in (A - B) \cap B$

$\Rightarrow x \in (A - B)$ and $x \in B$

$\Rightarrow (x \in A$ and $x \notin B)$ and $x \in B$

$\Rightarrow x \in A$ and $(x \notin B$ and $x \in B)$

But it is impossible that $x \in B$ and $x \notin B$ simultaneously.

$\therefore$ Our assumption is wrong, i.e., $(A - B) \cap B \neq \phi$ is a wrong statement.

Hence $\quad (A - B) \cap B = \phi$.

(iii) Let $x \in (A - B) \cup (B - A)$. Then

$x \in (A - B) \cup (B - A) \Rightarrow (x \in A$ and $x \notin B)$ or $(x \in B$ and $x \notin A)$

$\Rightarrow (x \in A$ or $x \in B)$ and $(x \notin B$ or $x \notin A)$

$\Rightarrow x \in (A \cup B)$ and $x \notin (A \cap B)$

$\Rightarrow x \in (A \cup B) - (A \cap B)$

$\therefore \quad (A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$ ...(3)

Similarly, we can show that $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$ ...(4)

From (3) and (4), we get $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$.

41. Let x be any element of $A-(B \cup C)$. Then

$$\begin{aligned} x \in A-(B \cup C) &\Rightarrow x \in A \text{ and } x \notin B \cup C \\ &\Rightarrow x \in A \text{ and } (x \notin B \text{ and } x \notin C) \\ &\Rightarrow (x \in A \text{ and } x \notin B) \text{ and } (x \in A \text{ and } x \notin C) \\ &\Rightarrow x \in A-B \text{ and } x \in A-C \\ &\Rightarrow x \in (A-B) \cap (A-C) \end{aligned}$$

$\therefore \quad A-(B \cup C) \subseteq (A-B) \cap (A-C)$ ...(1)

Now, let x be any element of $(A-B) \cap (A-C)$. Then

$$\begin{aligned} x \in (A-B) \cap (A-C) &\Rightarrow x \in A-B \text{ and } x \in A-C \\ &\Rightarrow (x \in A \text{ and } x \notin B) \text{ and } (x \in A \text{ and } x \notin C) \\ &\Rightarrow x \in A \text{ and } (x \notin B \text{ and } x \notin C) \\ &\Rightarrow x \in A \text{ and } x \notin B \cup C \\ &\Rightarrow x \in A-(B \cup C) \end{aligned}$$

$\therefore \quad (A-B) \cap (A-C) \subseteq A-(B \cup C)$ ...(2)

Combining (1) and (2), we have $A-(B \cup C)=(A-B) \cap (A-C)$.

42. Let x be any element of $A'-B'$. Then

$$\begin{aligned} x \in A'-B' &\Rightarrow x \in A' \text{ and } x \notin B' \Rightarrow x \notin A \text{ and } x \in B \\ &\Rightarrow x \in B \text{ and } x \notin A \Rightarrow x \in B-A \end{aligned}$$

$\therefore \quad A'-B' \subseteq B-A$ ...(1)

Now, let x be any element of $B-A$. Then

$$\begin{aligned} x \in B-A &\Rightarrow x \in B \text{ and } x \notin A \Rightarrow x \notin B' \text{ and } x \in A' \\ &\Rightarrow x \in A' \text{ and } x \notin B' \Rightarrow x \in A'-B' \end{aligned}$$

$\therefore \quad B-A \subseteq A'-B'$ ...(2)

Combining (1) and (2), we get $A'-B'=B-A$.

APPLICATIONS OF SETS

Uptill now we were discussing in this chapter, few operations on an abstract set. We shall now discuss the practical applications of these operations of set theory.

To Find a Formula for $n(A \cup B)$

Let A and B be two sets. Then the following two cases arise:

Case I: A and B are disjoint sets, i.e., A and B have no common element.

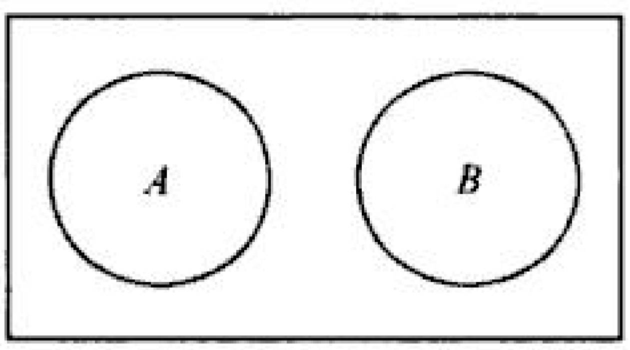

Fig. 1.22

$$\therefore \quad n(A \cup B)=n(A)+n(B)$$

Note: The number of elements in a set A is called *cardinal number of* A and is denoted by $n(A)$.

Case II: A and B are not disjoint sets.

(i) From Venn diagram, [Fig. 1.23] it is clear that

$$n(A \cup B)=n(A)+n(B)-n(A \cap B)$$

(ii) From Venn diagram, [Fig. 1.23] it is clear that

$$A \cup B=(A-B) \cup (A \cap B) \cup (B-A)$$

Now, as $(A-B)$, $(A \cap B)$ and $(B-A)$ are all disjoint sets.

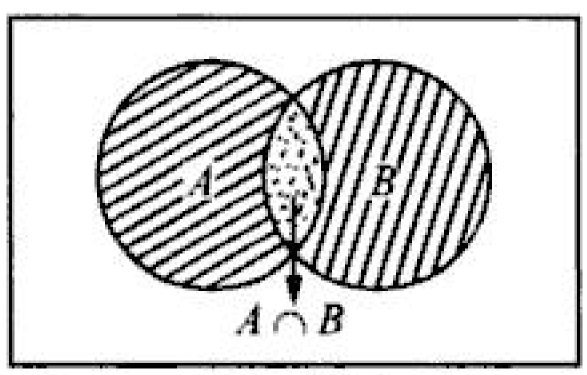

Fig. 1.23

$\therefore$ By case I,

$n(A \cup B) = n(A - B) + n(A \cap B) + n(B - A)$

(iii) From Venn diagram, [Fig. 1.24] it is clear that

$$A = (A - B) \cup (A \cap B)$$

$\therefore$ $\quad n(A) = n(A - B) + n(A \cap B)$

Similarly, $\quad n(B) = n(B - A) + n(A \cap B)$

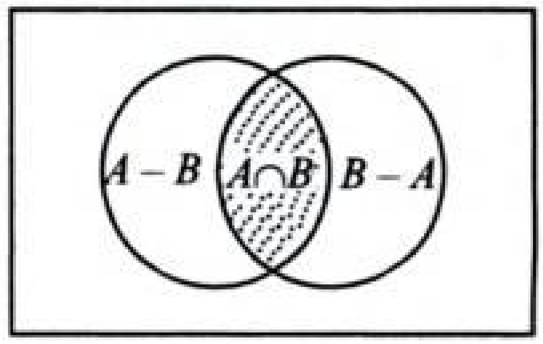

Fig. 1.24

Cor. $\quad n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C)$
$\quad -n(C \cap A) + n(A \cap B \cap C)$

Proof. Let $B \cup C = X$

$\therefore$ L.H.S $= n(A \cup X) = n(A) + n(X) - n(A \cap X)$

$= n(A) + n(B \cup C) - n[A \cap (B \cup C)]$

$= n(A) + n(B) + n(C) - n(B \cap C) - n[(A \cap B) \cup (A \cap C)]$

$= n(A) + n(B) + n(C) - n(B \cap C) - [n(A \cap B) + n(A \cap C)$
$- n\{(A \cap B) \cap (A \cap C)\}]$

$= n(A) + n(B) + n(C) - n(B \cap C) - n(A \cap B)$
$- n(A \cap C) + n\{(A \cap B) \cap (A \cap C)\}$

$= n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C)$
$- n(A \cap C) + n(A \cap B \cap C) =$ R.H.S.

Note: Let A, B and C be finite sets. Then

- $n(A - B) = n(A) - n(A \cap B)$
- $n(B - A) = n(B) - n(A \cap B)$
- No. of elements in exactly two of the sets A, B and C
  $= n(A \cap B \cap C') + n(A \cap B' \cap C) + n(A' \cap B \cap C)$
  $= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$
- No. of elements in exactly one of sets A, B and C
  $= n(A \cap B' \cap C') + n(A' \cap B \cap C') + n(A' \cap B' \cap C)$
  $= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(C \cap A) + 3n(A \cap B \cap C)$
- $n(A' \cup B') = n(A \cap B)') = n(U) - n(A \cap B)$
- $n(A' \cap B') = n((A \cup B)') = n(U) - n(A \cup B)$

Example 1. *X and Y are two sets such that $n(X) = 17$, $n(Y) = 23$ and $n(X \cup Y) = 38$. find $n(X \cap Y)$.*

Solution: $n(X) = 17$, $n(Y) = 23$, $n(X \cup Y) = 38$, $n(X \cap Y) = ?$

Now, $\quad n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$.

Then

$$38 = 17 + 23 - n(X \cap Y) \Rightarrow n(X \cap Y) = 17 + 23 - 38 = 2$$

Example 2. (i) *If X and Y are two sets such that $X \cup Y$ has 18 elements, X has 8 elements, and Y has 15 elements, how many elements does $X \cap Y$ have?*

(ii) *If A and B are two sets such that A has 40 elements, $A \cup B$ has 60 elements and $A \cap B$ has 10 elements, how many elements does B have?*

Solution: We are given $n(X \cup Y) = 18$, $n(X) = 8$, $n(Y) = 15$. Using the formula

$$n(X \cap Y) = n(X) + n(Y) - n(X \cup Y)$$

we get

$$n(X \cap Y) = 8 + 15 - 18 = 5$$

(ii) We are given $n(A) = 40$, $n(A \cap B) = 60$ and $n(A \cap B) = 10$. Putting these values in the formula $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, we get

$$60 = 40 + n(B) - 10 \Rightarrow n(B) = 30$$

Example 3. *Let $n(U) = 700$, $n(A) = 200$, $n(B) = 300$, $n(A \cap B) = 100$. Find $n(A' \cap B')$.*

Solution: $n(A' \cap B') = n(A \cup B)' = n(U) - n(A \cup B) = 700 - n(A \cup B)$

Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B) = 200 + 300 - 100 = 400$

Hence $n(A' \cup B') = 700 - 400 = 300$

Example 4. *If S and T are two sets such that S has 21 elements, T has 32 elements, and $S \cap T$ has 11 elements, how many elements does $S \cup T$ have?*

Solution: $n(S) = 21$, $n(T) = 32$, $n(S \cap T) = 11$, $n(S \cup T) = ?$

Using $n(S \cup T) = n(S) + n(T) - n(S \cap T) = 21 + 32 - 11 = 42$

Hence $S \cup T$ has 42 elements.

Example 5. *(i) In a group of 1000 people, there are 750 people who can speak Hindi and 400 who can speak English. How many can speak Hindi only?*

(ii) In a group of 400 people, 250 can speak Hindi and 200 can speak English. How many people can speak both Hindi and English?

Solution: (i) Refer to Fig. 1.25.

Here $n(H \cup E) = 1000$, $n(H) = 750$, $n(E) = 400$

Using $n(H \cup E) = n(H) + n(E) - n(H \cap E)$

we get $1000 = 750 + 400 - n(H \cap E)$

$\Rightarrow$ $n(H \cap E) = 1150 - 1000 = 150$

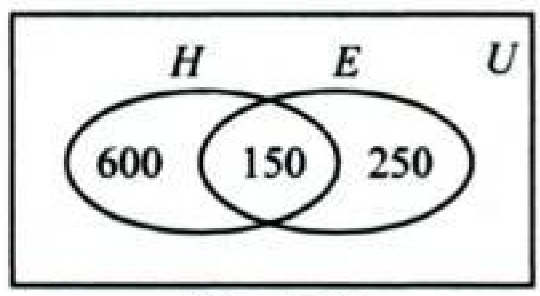

Fig. 1.25

Number of people who can speak Hindi only

$$= n(H \cap E') = n(H) - n(H \cap E) = 750 - 150 = 600$$

(ii) Let H and E denote the sets of people who can speak Hindi and English respectively.

Then, $n(H) = 250$, $n(E) = 200$ and $n(H \cup E) = 400$.

Now, $n(H \cap E) = n(H) + n(E) - n(H \cap E)$

$= 250 + 200 - 400 = 50$

Thus, 50 people can speak both Hindi and English.

Example 6. *In a school with 727 students, 600 students offer Mathematics and 173 students offer both Mathematics and Physics. How many students are enrolled in:*

(i) *Physics* (ii) *Physics only?*

Solution: Let M be the set of students offering Mathematics and P the set of students offering Physics (Fig. 1.26).

We are given that $n(M \cup P) = 727$, $n(M) = 600$,

$n(M \cap P) = 173$.

Using $n(M \cup P) = n(M) + n(P) - n(M \cap P)$

We get $727 = 600 + n(P) - 173$

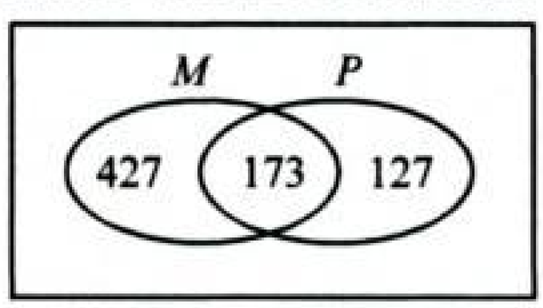

Fig. 1.26

$\Rightarrow \qquad 727 = 427 + n(P)$

$\Rightarrow \qquad n(P) = 727 - 427 = 300.$

$\therefore$ No. of students enrolled in Physics $= n(P) = 300$

No. of students enrolled in Physics only $= n(P \cap M')$

$= n(P) - n(P \cap M)$

$= 300 - 173 = 127$

Example 7. *Out of 20 members in a family, 12 like to take tea and 15 like coffee. Assume that each one likes at least one of the two drinks, how many like*

(i) *Only tea and not coffee?* (ii) *only coffee and not tea?*

(iii) *both coffee and tea?*

Solution: Let T be the set of people who like tea and C be the set of people who like coffee.

$\therefore \qquad n(T) = 12, n(C) = 15$ and $n(T \cup C) = 20$

Using $\qquad n(T \cup C) = n(T) + n(C) - n(T \cap C)$

We get $\qquad 20 = 12 + 15 - n(T \cap C) \Rightarrow n(T \cap C) = 27 - 20 = 7$

Hence 7 people like to take both coffee and tea.

Also $\qquad n(T \cap C') = n(T) - n(T \cap C) = 12 - 7 = 5$

i.e., 5 people like to take only tea and not coffee.

Again $\qquad n(C \cap T') = n(C) - (T \cap C) = 15 - 7 = 8$

i.e., 8 people like to take only coffee and not tea.

Example 8. *In a class of 50 students, 35 opted for Mathematics and 37 opted for Biology. How many have opted for both Mathematics and Biology? How many have opted for only Mathematics? (Assume that each student has to opt for at least one of the subjects.)*

Solution: Here $n(M \cup B) = 50, n(M) = 35, n(B) = 37, n(M \cap B) = ?$

Using $\qquad n(M \cap B) = n(M) + n(B) - n(M \cap B)$

We get $\qquad 50 = 35 + 37 - n(M \cap B)$

$\Rightarrow \qquad n(M \cap B) = 35 + 37 - 50 = 72 - 50 = 22$

$\therefore$ 22 students have opted for both Mathematics and Biology.

Again number of students who have opted for only mathematics

$= n(M) - n(M \cap B) = 35 - 22 = 13$

Example 9. *(i) In a group of 70 people, 37 like coffee, 52 like tea and each person likes at least one of the two drinks. How many people like both coffee and tea?*

(ii) In a group of 65 people, 40 like cricket, 10 like both cricket and tennis. How many like tennis only and not cricket? How many like tennis?

Solution: Let $\qquad A$ = set of people who like coffee

and $\qquad B$ = set of people who like tea

Then $\qquad A \cup B$ = set of people who like at least one of the two drinks

and $\qquad A \cap B$ = set of people who like both the drinks

Here $\qquad n(A) = 37, n(B) = 52, n(A \cup B) = 70$. Using the result,

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

We have $70 = 37 + 52 - n(A \cap B)$

$\Rightarrow$ $n(A \cap B) = 89 - 70 = 19$

$\therefore$ 19 people like both coffee and tea.

(ii) Let C be the set of people who like cricket and T be the set of people who like tennis.

Then $n(C \cup T) = 65, n(C) = 40, n(C \cap T) = 10$

We know that

$$n(C \cup T) = n(C) + n(T) - n(C \cap T)$$
$$\Rightarrow 65 = 40 + n(T) - 10$$
$$\Rightarrow n(T) = 65 - 40 + 10 = 35$$

Number of people who like only tennis

$= n(T) - n(C \cap T)$

$= 35 - 10 = 25$

$\therefore$ Number of people who like tennis only and not cricket = 25, and number of people who like tennis is 35.

Example 10. *In a group of 70 people, 45 speak Hindi language and 33 speak English language and 10 speak neither Hindi nor English. How many can speak both English as well as Hindi language? How many can speak only English language?*

Solution: Refer to Fig. 1.27. Here $n(H \cup E) = n(U) - n(H \cup E)' = 70 - 10 = 60$.

Using $n(H \cup E) = n(H) + n(E) - n(H \cap E)$

We have $60 = 45 + 33 - n(H \cap E)$

$\therefore$ $n(H \cap E) = 78 - 60 = 18$

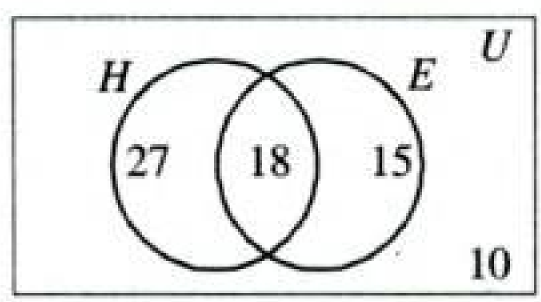

Fig. 1.27

Hence number of students who speak both English as well as Hindi are 18.

Again $A(E \cap H') = n(E) - n(H \cap E) = 33 - 18 = 15$

Hence number of students who speak only English are 15.

Example 11. *The members of a group of 400 people speak either Hindi or English or both, if 270 speak Hindi only and 50 speak both Hindi and English, how many of them speak English only?*

Solution: Refer to Fig. 1.28. Let x be the number of people who speak English only.

$n(H \cup E) = 400, n(H \cap E) = 50$

Then from the Venn diagram,

$400 = 270 + 50 + x$

$\therefore$ $x = 400 - 270 - 50 = 80$

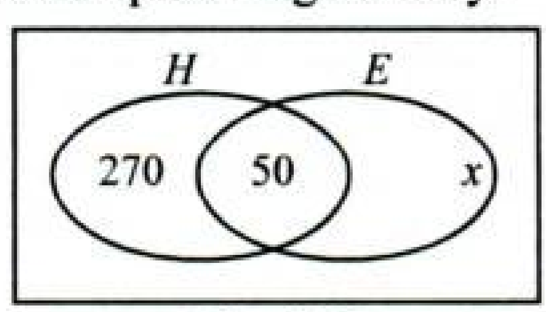

Fig. 1.28

We can present the solution in the following way also:

$n(H \cup E) = 400, n(H \cap E') = 270, n(H \cap E) = 50$

Using $n(H \cup E) = n(H \cap E') + n(E \cap H') + n(H \cap E)$

We get $400 = 270 + n(E \cap H') + 50$

$\Rightarrow$ $400 = 320 + n(E \cap H')$

or $n(E \cap H') = 400 - 320 = 80$

∴ Number of members who speak English only (i.e., who speak English, but not Hindi)

$$= n(E \cap H') = 80$$

Example12. *In a town with a population of* 5000, 3200 *people are egg-eaters,* 2500 *meat eaters and* 1500 *eat both egg and meat. How many are pure vegetarians?*

Solution: Let E be the set of people who are egg-eaters and M be the set of people who are meat-eaters (Fig. 1.29). We have $n(E) = 3200$, $n(M) = 2500$, $n(E \cap M) = 1500$.

Using $\quad n(E \cup M) = n(E) + n(M) - n(E \cap M)$

$$= 3200 + 2500 - 1500$$

$$= 5700 - 1500 = 4200$$

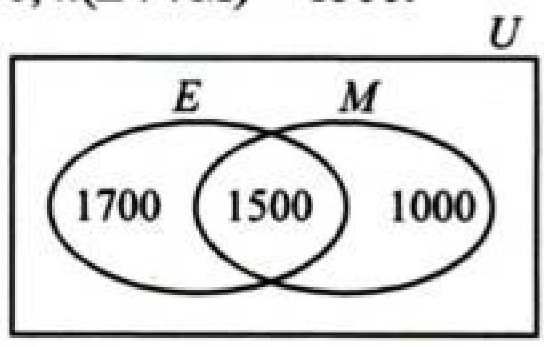

Fig. 1.29

∴ Number of pure vegetarians

$$= n(U) - n(E \cup M) = 5000 - 4200 = 800$$

Example 13. *Use a Venn diagram to solve the following problem. In a statistical investigation of* 1,003 *families of Kolkata, it was found that* 63 *families had neither a radio nor a TV,* 794 *families had a radio and* 187 *had a TV. How many families in that group had both a radio and a TV?*

Solution: Let R be the set of families having a radio and T the set of families having a TV. Then

$n(R \cup T)$= the no. of families having at least one of the radio and TV

$= 1,003 - 63 = 940$

$n(R) = 794$ and $n(T) = 187$

Let x families had both a radio and a TV, i.e., $n(R \cap T) = x$

The no. of families who have only Radio $= 794 - x$,

and the no. of families who have only TV $= 187 - x$

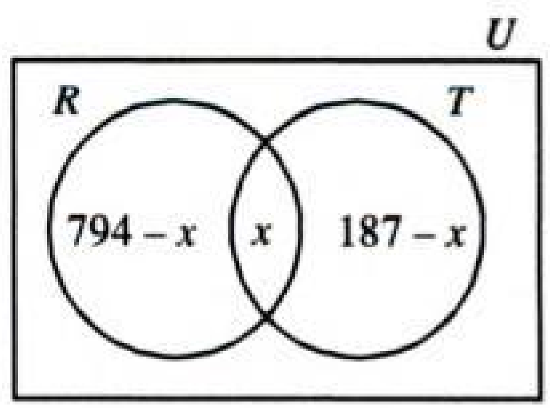

Fig. 1.30

From Venn diagram,

$$794 - x + x + 187 - x = 940$$

$\Rightarrow 981 - x = 940 \quad$ or $\quad x = 981 - 940 = 41$

Hence the required no. of families having both a radio and a TV = 41

Example 14. *Every student in a class of* 42 *students, studies at least one of the subjects, Mathematics, English and Commerce,* 14 *students study Mathematics,* 20 *Commerce and* 24 *English.* 3 *students study Mathematics and Commerce,* 2 *English and Commerce and there is no student who studies all the three subjects. Find the number of students who study Mathematics, but not Commerce.*

Solution: Let M be the set of students who study Mathematics (Fig. 1.31). Similarly, the set, E and C for students who study English and Commerce respectively. Then

$n(M \cup E \cup C) = 42$, $n(M) = 14$, $n(E) = 24$, $n(C) = 20$,

$n(M \cap E) = 3$, $n(E \cap C) = 2$, $n(M \cap E \cap C) = 0$

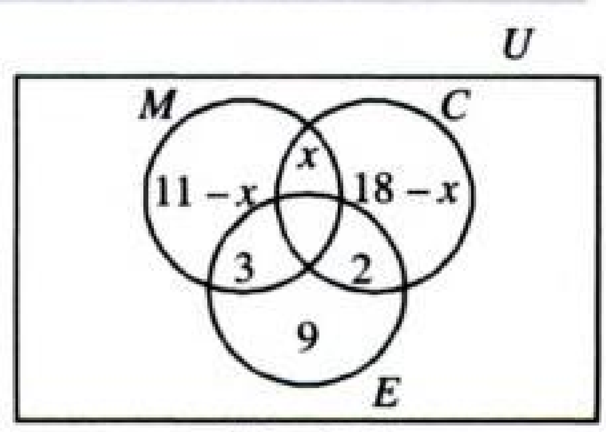

Fig. 1.31

Let x be the number of students who study Mathematics and Commerce, i.e., $n(M \cap C) = x$.

Then

No. of students who study only Mathematics = $14 - (x + 3) = 11 - x$

No. of students who study only Commerce = $20 - (x + 2) = 18 - x$

No. of students who study only English = $24 - (3 + 2) = 19$

From the Venn diagram, we have

$$11 - x + x + 18 - x + 2 + 3 + 19 = 42$$

$$\Rightarrow \quad 53 - x = 42; x = 53 - 42 = 11$$

Hence the reqd. no of students who study Mathematics, but not Commerce

$$= 14 - x = 14 - 11 = 3$$

Example 15. *Of the members of three athletic teams in a certain school,* 21 *are on the basket ball team,* 26 *on the hockey team, and* 29 *on the football team,* 14 *play hockey and basketball,* 15 *play hockey and football, and* 12 *play football and basketball.* 8 *are on all the three teams? How many members are there altogether?*

Solution: Let B, H, F denote the sets of members who are on the basketball team, hockey team and football team respectively (Fig. 1.32).

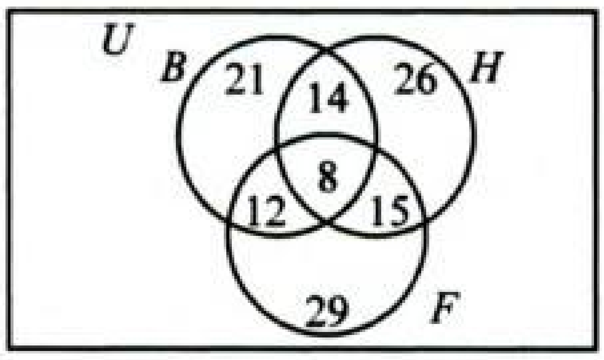

Fig. 1.32

Then $n(B) = 21, n(H) = 26, n(F) = 29, n(H \cap B) = 14$, $n(H \cap F) = 15$, $n(F \cap B) = 12$ and $n(B \cap H \cap F) = 8$

We have to find $n(B \cup H \cup F)$.

Using the result,

$$n(B \cup H \cup F) = n(B) + n(H) + n(F) - n(B \cap H) - n(H \cap F) - n(F \cap B) + n(B \cap H \cap F)$$

We get $\quad n(B \cup H \cup F) = 21 + 26 + 29 - 14 - 15 - 12 + 8 = 84 - 41 = 43$

Hence there are 43 members altogether.

Example 16. *In a group of* 100 *people,* 65 *like to play cricket,* 40 *like to play tennis and* 55 *like to play volleyball. All of them like to play at least one of the three games. If* 25 *like to play both cricket and tennis,* 24 *like to play both tennis and volleyball and* 22 *like to play both cricket and volleyball, then*

(i) *how many like to play all the three games?*

(ii) *how many like to play cricket only?*

(iii) *how many like to play tennis only?*

Represent above information in a Venn diagram.

Solution: Refer to Fig. 1.33. Let $n(C)$ represent the number of people playing cricket, $n(T)$ represent the number of people playing tennis and $n(V)$ represent the number of people playing volley ball. Then

$n(C \cup T \cup V) = 100, n(C) = 65, n(T) = 40,$

$n(V) = 55, n(C \cap T) = 25,$

$n(T \cap V) = 24, n(C \cap V) = 22$

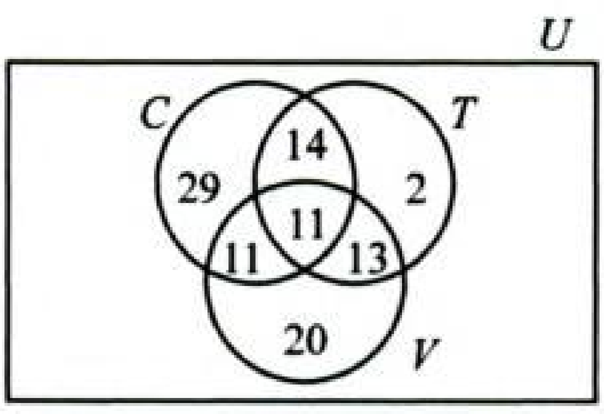

Fig. 1.33

Using $n(C \cup T \cup V) = [n(C) + n(T) + n(V)] - [n(C \cap T) + n(T \cap V) + n(C \cap V)] + n(C \cap T \cap V)$

$$\Rightarrow \quad n(C \cap T \cap V) = n(C \cup T \cup V) - [n(C) + n(T) + n(V)]$$
$$+ [n(C \cap T) + n(T \cap V) + n(C \cap V)]$$
$$= 100 - (65 + 40 + 55) + (25 + 24 + 22)$$
$$= 100 - 160 + 71$$
$$= 11$$

Thus, (i) 11 people like to play all the three games.

(ii) The number of people who like to play cricket only
$$= 65 - (14 + 11 + 11) = 65 - 36 = 29$$

(iii) The number of people who like to play tennis only
$$= 40 - (14 + 11 + 13) = 40 - 38 = 2$$

Example 17. *In a class,* 22 *students offered Mathematics,* 18 *students offered Chemistry and* 24 *students offered Physics. All of them have to offer atleast one of the three subjects. Of these,* 11 *are in both Mathematics and Chemistry,* 13 *in Chemistry and Physics and* 7 *have offered all the three subjects. Find*

(i) *how many students are there in the class?*
(ii) *how many students offered only Mathematics?*

Solution: Refer to Fig. 1.34. Here, we have

$$n(M) = 22, \; n(C) = 18, \; n(P) = 24$$
$$n(M \cap C) = 11, \; n(C \cap P) = 13,$$
$$n(M \cap P) = 14, \; n(M \cap P \cap C) = 7$$

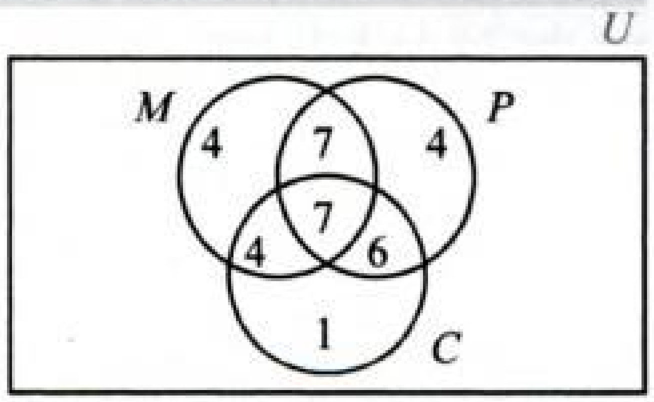

Fig. 1.34

Using

$$n(M \cup P \cup C) = [n(M) + n(P) + n(C)] - [n(M \cap P)$$
$$+ n(M \cap C) + n(P \cap C)] + n(M \cap P \cap C)$$
$$= (22 + 24 + 18) - (14 + 11 + 13) + 7 = 64 - 38 + 7 = 33$$

Thus, there are 33 students in the class.

(ii) The number of students who offered only mathematics
$$= 22 - (7 + 7 + 4) = 22 - 18 = 4$$

Example 18. *In a class of* 140 *students,* 60 *play football,* 48 *play hockey and* 75 *play cricket,* 30 *play hockey and cricket,* 18 *play football and cricket,* 42 *play football and hockey and* 8 *play all the three games. Use Venn diagram to find*

(i) *students who do not play any of these three games;*
(ii) *students who play only cricket;*
(iii) *students who play football and hockey, but not cricket.*

Solution: Refer to Fig. 1.35. We have

$$n(U) = 140, \; n(F) = 60, \; n(H) = 48, \; n(C) = 75$$
$$n(F \cap H) = 42, \; n(F \cap C) = 18, \; n(H \cap C) = 30$$
$$n(F \cap H \cap C) = 8$$
$$\therefore n(F \cup H \cup C) = n(F) + n(H) + n(C)$$
$$+ n(F \cap H \cap C) - n(F \cap C)$$
$$- n(H \cap C) - n(F \cap H)$$
$$= 60 + 48 + 75 + 8 - 42 - 18 - 30 = 191 - 90 = 101$$

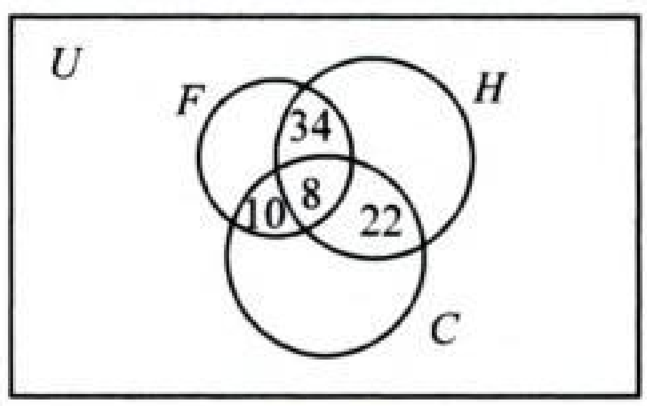

Fig. 1.35

(i) Students who do not play any of three games

$$= n(U) - n(F \cup H \cup C) = 140 - 101 = 39$$

(ii) Students who play only cricket

$$= n(C) - (10 + 8 + 22) = 75 - 40 = 35$$

(iii) Students who play football and hockey, but not cricket

$$= n(F \cap H) - n(F \cap H \cap C) = 42 - 8 = 34$$

Example 19. *In a city, three daily newspapers A, B and C are published. 42% of the people on that city read A, 51% read B, 68% read C; 30% read A and B, 28% read B and C, 36% read A and C, 8% do not read any of the three newspapers. Find the percentage of persons who read all the three papers.*

Solution: Let the number of persons in the city be 100.

Then, we have

$$n(A) = 42, n(B) = 51, n(C) = 68$$

$$n(A \cap B) = 30, n(B \cap C) = 28, n(A \cap C) = 36$$

$$n(A \cup B \cup C) = 100 - 8 = 92$$

Using $$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

Substituting the above values, we have

$$92 = 42 + 51 + 68 - 30 - 28 - 36 + n(A \cap B \cap C) \Rightarrow n(A \cap B \cap C) = 92 - 161 + 94$$

$$\Rightarrow n(A \cap B \cap C) = 92 - 67 = 25$$

Hence 25% of the people read all the three papers.

Example 20. *If A and B be two sets containing 3 and 6 distinct elements respectively, what can be the minimum number of elements in $A \cup B$? Find also the maximum number of elements in $A \cup B$.*

Solution: We know that $n(A \cup B) = n(A) + n(B) - n(A \cap B)$. Clearly $n(A \cup B)$ is minimum when $n(A \cap B)$ is maximum and $n(A \cup B)$ is maximum when $n(A \cap B)$ is minimum. Now, the maximum value of $n(A \cap B)$ is 3 and this occurs when $A \subset B$. Therefore,

∴ The minimum number of elements in

$$A \cup B = n(A) + n(B) - n(A \cap B) = 3 + 6 - 3 = 6$$

Also the minimum number of elements in $(A \cap B)$ is 0 and this occurs when $A \cap B = \phi$, i.e., when A and B are disjoint sets. Therefore, The maximum number of elements in

$$A \cup B = n(A) + n(B) - n(A \cap B) = 3 + 6 - 0 = 9$$

Example 21. *A market research group conducted a survey of 1000 consumers and reported that 720 consumers liked product A and 450 consumers liked product B. What is the least number that must have liked both products?*

Solution: Let U be the set of consumers questioned. S be the set of consumers that liked the product A and T be the set of consumers who liked the product B.
Given: $n(U) = 1000$, $n(S) = 720$, $n(T) = 450$. So,

$$n(S \cup T) = n(S) + n(T) - n(S \cap T) = 1170 - n(S \cap T)$$

Therefore, $n(S \cap T)$ is least when $n(S \cup T)$ is maximum.

But $S \cup T \subset U$ implies $n(S \cup T) \leq n(U) = 1000$. So, maximum values of $n(S \cup T)$ is 1000. Thus, the least value of $n(S \cap T)$ is 170.
Hence, the least number of consumers who liked both products is 170.

Example 22. *Out of* 500 *car owners investigated,* 400 *owned car A and* 200 *owned car B,* 50 *owned both A and B cars. Is this data correct?*

Solution: Let U be the set of car owners investigated. M be the set of persons who owned car A and S be the set of persons who owned car B. Given that $n(U) = 500$, $n(M) = 400$, $n(S) = 200$ and $n(S \cap M) = 50$.

Then $$n(S \cup M) = n(S) + n(M) - n(S \cap M) = 550.$$

But $S \cup M \subset U$ implies
$$n(S \cup M) \leq n(U) = 500.$$
This is a contradiction. So, the given data is incorrect.

EXERCISE 1.5

LEVEL OF DIFFICULTY A

1. Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6, 8\}$, verify the formula:
$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
2. If A and B are two sets such that A has 12 elements. B has 17 elements, and $A \cup B$ has 21 elements, how many elements does $A \cap B$ have?
3. If X and Y are two sets such that $X \cup Y$ has 18 elements, X has 8 elements and Y has 15 elements, how many elements does $X \cap Y$ have?
4. X and Y are two sets such that X has 40 elements, $X \cup Y$ has 60 elements, and $X \cap Y$ has 10 elements. How many elements does Y have?
5. In a class of 60 boys, there are 45 boys who play cards and 30 boys who play carrom. Use set operations to show:
   (i) How many boys play both the games? (ii) How many play cards only?
   (iii) How many play carrom only?
6. In a group of 400 people, 250 can speak English only and 70 can speak Hindi only. Find:
   (i) How many can speak English? (ii) How many can speak Hindi?
   (iii) How many can speak both English and Hindi?
7. There are 210 members in a club, 100 of them drink tea and 65 drink tea, but not coffee. Find:
   (i) How many drink coffee? (ii) How many drink coffee, but not tea?
8. In a class of 25 students, 12 students have taken Economics, 8 have taken Economics, but not Maths. Find (i) the number of students who have taken Economics and Maths; (ii) those who have taken Maths, but not Economics.
9. In a group of athletic teams in a school, 21 are in the basket ball team; 26 in the hockey team and 29 in football team. If 14 play hockey and basket ball; 12 play football and basket ball; 15 play hockey and football and 8 play all the three games. Find:
   (i) how many players are there in all? (ii) how many play football only?
10. In a committee 50 people speak French, 20 speak Spanish and 10 speak both Spanish and French. How many speak atleast one of these two languages?
11. A class has two teams, a football and a cricket team containing 9 and 11 players respectively. If the total number of students in these teams is 15, find the number of students playing for both the teams.

12. In a group of 26 persons, 8 take tea, but not coffee and 16 take tea. How many take coffee, but not tea?

13. In a survey of 60 people, it was found that 25 people read Newspaper H, 26 read Newspaper T, 26 read Newspaper I, 9 read both H and I, 11 read both H and T, 8 read both T and I, 3 read all three newspapers. Find:

(i) the number of people who read at least one of the newspapers;
(ii) the number of people who read exactly one newspaper.

14. In a survey of 600 students in a college, 150 were listed as drinking tea, 225 as drinking coffee and 100 were listed as both drinking tea as well as coffee. Find how many students were drinking neither tea nor coffee.

15. In a class of 35 students, 17 can speak Arabic, 10 can speak Arabic, but not Bangla. Find the number of students who can speak both Arabic and Bangla and the number of students who can speak Bangla, but not Arabic if it is given that each student can speak either Arabic or Bangla or both.

16. In a group of students, 100 students know Hindi, 50 know English and 25 know both. Each of the students knows either Hindi or English. How many students are there in the group?

17. In a survey, it is found that 21 people like product A, 26 like product B and 29 like product C. If 14 people like products A and B, 12 people like products C and A, 14 people like products B and C and 8 like all the three products. Find how many like product C only.

LEVEL OF DIFFICULTY B

18. A college awarded 38 medals in football, 15 in basketball and 20 to cricket. If these medals went to a total of 58 men and only three men got medals in all the three sports, how many received medals in exactly two of the three sports?

19. Out of 1020 boys in a school, 406 play cricket, 324 play hockey and 250 play football. 80 boys play cricket and hockey, 64 play hockey and football, 92 play football and cricket while 30 play all the three games. How many boys play none of the games?

20. In a survey of 25 students, it was found that 15 had taken mathematics, 12 had taken physics and 11 had taken chemistry, 5 had taken mathematics and chemistry, 9 had taken mathematics and physics, 4 had taken physics and chemistry and 3 had taken all the three subjects. Find the number of students that had

(i) only chemistry; (ii) only mathematics; (iii) only physics;
(iv) physics and chemistry, but not mathematics;
(v) mathematics and physics but not chemistry;
(vi) only one of the subjects;
(vii) at least one of the three subjects; (viii) none of the subjects.

21. A market research group conducted a survey of 2000 consumers and reported that 1720 consumers liked product A and 1450 consumers liked product B. What is the least number that must have liked both the products?

22. In a survey of 100 students, the number of students studying the various languages were found to be: English only 18, English but not Hindi 23, English and Sanskrit 8, English 26, Sanskrit 48, Sanskrit and Hindi 8, no language 24. Find:

(i) How many students were studying Hindi?
(ii) How many students were studying English and Hindi?

23. In a survey of 100 persons it was found that 28 read magazine A, 30 read magazine B, 42 read magazine C, 8 read magazines A and B, 10 read magazines A and C, 5 read magazines B and C and 3 read all the three magazines. Find:

(i) How many read none of the three magazines?
(ii) How many read magazine C only?

24. In an examination, 80 students secured first class marks in English or Mathematics. Out of these 50 students obtained first class marks in Mathematics and 10 students in English and Mathematics both. How many students secured first class marks in English only?

25. In an examination, 80% students passed in Mathematics, 72% passed in Science and 13% failed in both the subjects. If 312 students passed in both the subjects, find the total number of students who appeared in the examination.

Answers

2. 8. **3.** 5. **4.** 30 **5.** (i) 15; (ii) 30; (iii) 15.
6. (i) 330; (ii) 150; (iii) 80. **7.** (i) 145; (ii) 110.
8. (i) 4; (ii) 13. **9.** (i) 13; (ii) 10. **10.** 60. **11.** 5. **12.** 10.
13. 52, 30. **14.** 325. **15.** 7, 18 **16.** 125. **17.** 11.
18. 9. **19.** 246.
20. (i) 5; (ii) 4; (iii) 2; (iv) 1; (v) 6; (vi) 11; (vii) 23; (viii) 2.
21. 1170. **22.** (i) 18; (ii) 3. **23.** (i) 20; (ii) 30. **24.** 30. **25.** 480.

HINTS AND SOLUTIONS

12. Here $n(T \cup C) = 26$

$n(T \cup C') = 8$, $n(T) = 16$

Using $n(T \cap C') = n(T) - n(T \cap C)$

We get $8 = 16 - n(T \cap C)$

$\Rightarrow$ $n(T \cap C) = 16 - 8 = 8$

Using $n(T \cup C) = n(T \cap C') + n(C \cap T') + n(T \cap C)$

We get $26 = 8 + n(C \cap T') + 8 \Rightarrow 26 = 16 + n(C \cap T')$

$\Rightarrow n(C \cap T') = 26 - 16 = 10$

$\therefore$ Number of persons who take coffee, but not tea $= n(C \cap T') = 10$

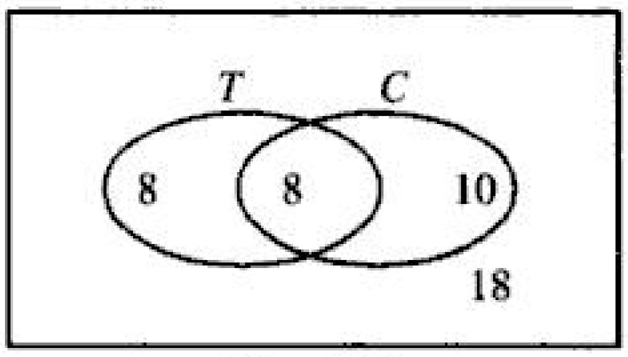

Fig. 1.36

13. We have $n(H) = 25$, $n(T) = 26$, $n(I) = 26$

$n(H \cap I) = 9$, $n(H \cap T) = 11$, $n(T \cap I) = 8$.

$n(H \cap T \cap I) = 3$

(i) The number of people who read at least one of the newspapers $= n(H \cup T \cup I)$

Now, $n(H \cup T \cup I) = n(H) + n(T) + n(I) - n(H \cap T) - n(H \cap I) - n(I \cap T) + n(H \cap T \cap I)$

$= 25 + 26 + 26 - 11 - 9 - 8 + 3$

$= 80 - 28$

$= 52$

(ii) The number of people who read exactly one newspaper

$= n(H) + n(T) + n(I) - 2\{n(H \cap T) + n(T \cap I) + n(H \cap I)\} + 3n(H \cap T \cap I)$

$= 25 + 26 + 26 - 2(11 + 8 + 9) + 3(3)$

$= 77 - 56 + 9$

$= 30$

14. Let U be the set of all surveyed students, A denote the set of students drinking tea and B be the set of students drinking coffee. It is given that $n(U) = 600$, $n(A) = 150$, $n(B) = 225$ and $n(A \cap B) = 100$. We have to find, $n(A' \cap B')$. Now,

$$\begin{aligned} n(A' \cap B') &= n(A \cup B)' = n(U) - n(A \cup B) \\ &= n(U) - \{n(A) + n(B) - n(A \cap B)\} \\ &= 600 - \{150 + 225 - 100\} \\ &= 600 - 275 \\ &= 325 \end{aligned}$$

15. $n(A \cup B) = 35$, $n(A) = 17$ and $n(A - B) = 10$. Now,

$$n(A) = n(A - B) + n(A \cap B) \Rightarrow 17 = 10 + n(A \cap B)$$
$$\Rightarrow n(A \cap B) = 7$$

Now, $n(A \cup B) = n(A) + n(B) - n(A \cap B) \Rightarrow 35 = 17 + n(B) - 7$
$$\Rightarrow n(B) = 25$$

Now, $n(B) = n(B - A) + n(A \cap B) \Rightarrow 25 = n(B - A) + 7$
$$\Rightarrow n(B - A) = 18$$

16. Here, $n(H) = 100$, $n(E) = 50$

and $n(H \cap E) = 25$

$n(H \cup E) = n(H) + n(E) - (H \cap E)$

$= 100 + 50 - 25 = 125$

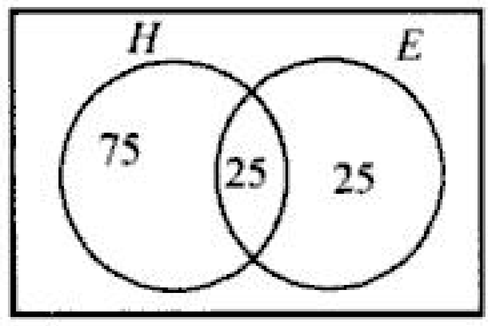

Fig. 1.37

17. Let A, B, C denote the sets of people who liked product A, product B and product C respectively (Fig. 1.38). Then, $n(A) = 21$, $n(B) = 26$, $n(C) = 29$, $n(A \cap B) = 14$, $n(C \cap A) = 12$, $n(B \cap C) = 14$ and $n(A \cap B \cap C) = 8$.

We have to find the number of people who liked product C only.

Clearly, from the Venn diagram, we get the required number as 11.

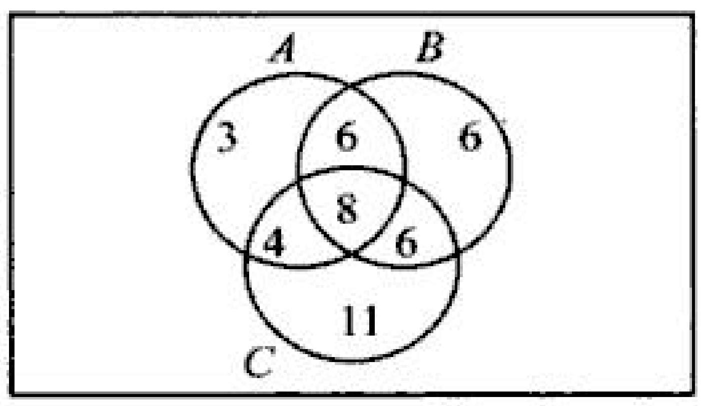

Fig. 1.38

18. Let F denote the set of men who received medals in football, B the set of men who received medals in basketball and C the set of men who received medals in cricket. Then, we have

$n(F) = 38$, $n(B) = 15$, $n(C) = 20$, $n(F \cup B \cup C) = 58$ and $n(F \cap B \cap C) = 3$

Now,

$$n(F \cup B \cup C) = n(F) + n(B) + n(C) - n(F \cap B) - n(B \cap C) - n(F \cap C) + n(F \cap B \cap C)$$

$$\Rightarrow 58 = 38 + 15 + 20 - n(F \cap B) - n(B \cap C) - n(F \cap C) + 3$$

$$\Rightarrow n(F \cap B) + n(B \cap C) + n(F \cap C) = 76 - 58 = 18$$

Now, Number of men who received medals in exactly two of the three sports

$$= n(F \cap B) + n(B \cap C) + n(F \cap C) - 3n(F \cap B \cap C) = 18 - 3 \times 3 = 9$$

Thus, 9 men received medals in exactly two of the three sports.

19. Let C = the set of the boys who play cricket
H = the set of boys who play hockey
F = the set of boys who play football

Then $n(C) = 406$, $n(H) = 324$, $n(F) = 250$, $n(C \cap H) = 80$, $n(H \cap F) = 64$, $n(F \cap C) = 92$, $n(C \cap H \cap F) = 30$

$\therefore\quad n(C \cup H \cup F) = n(C) + n(H) + n(F) - n(C \cap H)$

$= - n(H \cap F) - n(F \cap C) + n(C \cap H \cap F)$

$= 406 + 324 + 250 - 80 - 64 - 92 + 30$

$= 1010 - 236$

$= 774$

Therefore, 774 boys play at least one of the games. The number of boys who play none of the games = 1020 – 774 = 246.

20. Let M denote the set of students who had taken Mathematics, P the set of students who had taken Physics and C the set of students who had taken Chemistry. Then, we have $n(U) = 25$, $n(M) = 15$, $n(P) = 12$, $n(C) = 11$, $n(M \cap C) = 5$, $n(M \cap P) = 9$, $n(P \cap C) = 4$, $n(M \cap P \cap C) = 3$.

(i) Required number of students

$= n(M' \cap P' \cap C) = n((M \cup P)' \cap C)$

$= n(C) - n((M \cup P) \cap C) \qquad [\because n(A \cap B') = n(A) - n(A \cap B)]$

$= n(C) - n((M \cap C) \cup (P \cap C))$

$= n(C) - \{n(M \cap C) + n(P \cap C) - n(M \cap P \cap C)\}$

$= 11 - \{5 + 4 - 3\} = 5$

(ii) Required number of students

$= n(M \cap P' \cap C')$

$= n(M \cap (P \cup C)')$

$= n(M) - n(M \cap (P \cup C))$

$= n(M) - n((M \cap P) \cup (M \cap C))$

$= n(M) - \{n(M \cap P) + n(M \cap C) - n(M \cap P \cap C)\}$

$= 15 - \{9 + 5 - 3\}$

$= 4$

(iii) Required number of students

$= n(P \cap M' \cap C')$

$= n(P \cap (M \cup C)')$

$= n(P) - n(P \cap (M \cup C))$

$= n(P) - n((P \cap M) \cup (P \cap C))$

$= n(P) - \{n(P \cap M) + n(P \cap C) - n(P \cap M \cap C)\}$

$= 12 - \{9 + 4 - 3\}$

$= 2$

(iv) Required number of students

$= n(P \cap C \cap M')$

$= n(P \cap C) - n(P \cap C \cap M) \qquad [\because n(A \cap B') = n(A) - n(A \cap B)]$

$= 4 - 3 = 1$

(v) Required number of students

$= n(M \cap P \cap C') = n(M \cap P) - n(M \cap P \cap C) = 9 - 3 = 6.$

(vi) Required number of students

$= n(M) + n(P) + n(C) - 2\{n(M \cap P) + n(P \cap C) + n(M \cap C)\}$

$+ 3n(M \cap P \cap C)$

$= 15 + 12 + 11 - 2\{9 + 4 + 5\} + 3 \times 3$

$= 38 - 36 + 9$

$= 11$

(vii) Required number of students

$$= n(M \cup P \cup C)$$
$$= n(M) + n(P) + n(C) - n(M \cap P) - n(P \cap C) - n(M \cap C) + n(M \cap P \cap C)$$
$$= 15 + 12 + 11 - 9 - 4 - 5 + 3$$
$$= 23$$

(viii) Required number of students

$$= n(M' \cap P' \cap C')$$
$$= n(M \cup P \cup C)'$$
$$= n(U) - n(M \cup P \cup C)$$
$$= 25 - 23$$
$$= 2$$

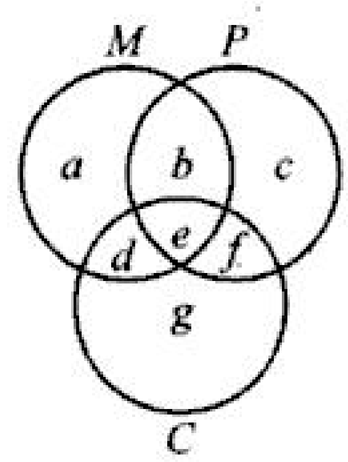

Fig. 1.39

Aliter. Consider the Venn diagram shown in Fig. 1.39. a, b, c, d, e, f, g denote the number of students in the respective regions.

From the Venn diagram, we have

$n(M) = a + b + d + e$, $n(P) = b + c + e + f$, $n(C) = d + e + f + g$, $n(M \cap P) = b + e$, $n(P \cap C) = e + f$, $n(M \cap C) = d + e$, and $n(M \cap P \cap C) = e$. It is given that

$$n(M \cap P \cap C) = 3 \Rightarrow e = 3$$
$$n(M \cap P) = 9 \Rightarrow b + e = 9 \Rightarrow b + 3 = 9 \Rightarrow b = 6$$
$$n(P \cap C) = 4 \Rightarrow e + f = 4 \Rightarrow 3 + f = 4 \Rightarrow f = 1$$
$$n(M \cap C) = 5 \Rightarrow d + e = 5 \Rightarrow d + 3 = 5 \Rightarrow d = 2$$
$$n(M) = 15 \Rightarrow a + b + d + e = 15 \Rightarrow a + 6 + 2 + 3 = 15 \Rightarrow a = 4$$
$$n(P) = 12 \Rightarrow b + c + e + f = 12 \Rightarrow 6 + c + 3 + 1 = 12 \Rightarrow c = 2$$
$$n(C) = 11 \Rightarrow d + e + f + g = 11 \Rightarrow 2 + 3 + 1 + g = 11 \Rightarrow g = 5$$

Now,

(i) Required number of students $= g = 5$

(ii) Required number of students $= a = 4$

(iii) Required number of students $= c = 2$

(iv) Required number of students $= f = 1$

(v) Required number of students $= b = 6$

(vi) Required number of students $= a + c + g = 4 + 2 + 5 = 11$

(vii) Required number of students $= a + b + c + d + e + f + g = 23$

(viii) Required number of students $= 25 - (a + b + c + d + e + f + g) = 25 - 23 = 2$

21. Given: $n(U) = 2000$, $n(A) = 1720$, $n(B) = 1450$

Then

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$n(A \cup B) = 1720 + 1450 - n(A \cap B)$$
$$= 3170 - n(A \cap B)$$

Since $A \cup B \subset U$, we have

$$\therefore \quad n(A \cup B) \leq n(U) \Rightarrow 3170 - n(A \cap B) \leq 2000$$
$$\Rightarrow 3170 - 2000 \leq n(A \cap B)$$
$$\Rightarrow n(A \cap B) \geq 1170$$

Thus, the least value of $n(A \cap B)$ is 1170.

22. Let U denote the set of surveyed students. Let E, H and S denote the set of student who are studying English, Hindi and Sanskrit respectively. Then, $n(U) = 100$, $n(E) = 26$, $n(S) = 48$, $n(E \cap S) = 8$, $n(S \cap H) = 8$, $n(E \cap H \cap S) = 3$.

Number of students who study English only = 18

Number of students who study no language = 24

Let us draw the Venn diagram as shown in Fig. 1.40.

Clearly, from the figure and with the given data, we have

Number of students who study Hindi only

$$= 100 - (18 + 5 + 3 + 5 + 35) - 24$$
$$= 100 - 66 - 24$$
$$= 100 - 90$$
$$= 10$$

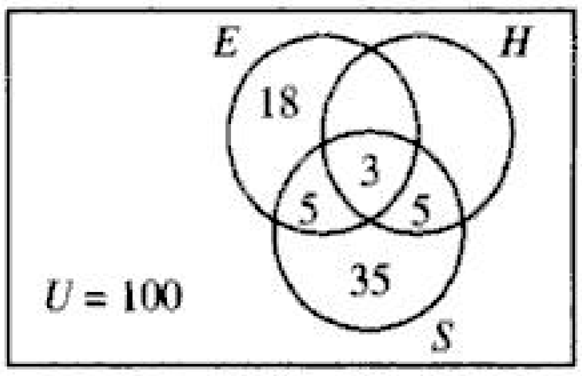

Fig. 1.40

$\therefore$ Number of students who study Hindi = 10 + 3 + 5 = 18

and the number of students who study English and Hindi = 3

23. Let U denote the set of surveyed persons. Let A, B and C denote the set of people who read magazine A, B and C respectively. As per data,

$$n(U) = 100,\ n(A) = 28,\ n(B) = 30,\ n(C) = 42,$$
$$n(A \cap B) = 8,\ n(A \cap C) = 10,\ n(B \cap C) = 5$$

and $\quad n(A \cap B \cap C) = 3$

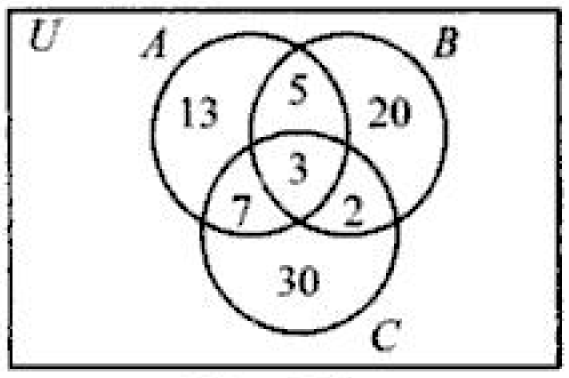

Fig. 1.41

Let us draw the Venn diagram as given in Fig. 1.41.

Since $n(A) + n(B) + n(C) = 28 + 30 + 42 = 100$

and $\quad n(U) = 100$

The number of persons who read none of the magazines is:

$$100 - (13 + 5 + 20 + 7 + 3 + 2 + 30) = 100 - 80 = 20$$

and number of persons who read magazine C only = 30

24. Let E be the set of students who secured first class marks in English and M the set of students who secured first class marks in Mathematics. The given information may be written as

$$n(E \cup M) = 80,\ n(M) = 50,\ n(E \cap M) = 10$$

We know that

$$n(E \cup M) = n(E) + n(M) - n(E \cap M)$$

Substituting the values of $n(E \cup M)$, $n(M)$ and $n(E \cap M)$, we get

$$80 = n(E) + 50 - 10$$

$\Rightarrow \quad n(E) = 80 - 40 = 40$

Thus, the number of students who secured first class marks in English only

$$= n(E) - n(E \cap M) = 40 - 10 = 30$$

25. As 80% students passed in Mathematics, therefore, the students who failed in Mathematics is 20%.

Also, 72% students passed in Science, therefore the students who failed in Science is 28%.

Since 13% students failed in both the subjects,

Students who failed in Mathematics only = 7%

and

Students who failed in Science only = 15%

$\therefore$ The students who failed in either of the two subjects or in both subjects

= 7% + 15% + 13% = 35%

The students who passed in both the subjects = 65%

Thus, if 65 students pass in both subjects, then the total number of students = 100,

$\therefore$ if 1 student passes in both subjects, then the total number of students = $\dfrac{100}{65}$,

$\therefore$ if 312 students pass in both subjects, then the total number of students

$$= \frac{100}{65} \times 312 = 480$$

CHAPTER

2

RELATIONS AND FUNCTIONS

Learning Objectives

After successful completion of this chapter, the reader should be able to understand and appreciate:

- Cartesian Product of Sets
- Relations
- Functions
- Some Important Functions

INTRODUCTION

In our day to day life, we observe many relations such as mother and daughter, father and son, and teacher and student. In mathematics also, we come across relations such as line l is perpendicular to line m, number a is greater than number b, and A is a subset of B. In all these a relation exists between pair of objects in certain order. This chapter is devoted to the study of several such relations and functions in mathematics.

ORDERED PAIR

Let A and B be two non-empty sets. If $a \in A$ and $b \in B$, then an element of the form (a, b) is called an *ordered pair*, where 'a' is regarded as '*the first element*' and 'b' as the *second element*.

It is evident from the definition that

(i) $(a, b) \neq (b, a)$

(ii) $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$

Equality of Two Ordered Pairs

Two ordered pairs (a, b) and (c, d) are said to be equal if and only if $a = c$ and $b = d$. The ordered pairs (2, 4) and (2, 4) are equal while the ordered pairs (2, 4) and (4, 2) are different. The distinction between the set {2, 4} and the ordered pair (2, 4) must be noted carefully. We have {2, 4} = {4, 2}, but (2, 4) ≠ (4, 2).

CARTESIAN PRODUCT OF SETS

Let A and B be two non-empty sets. The cartesian product of A and B is denoted by $A \times B$ (read as 'A cross B') and is defined as the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Symbolically,

$$A \times B = \{(a, b): a \in A \text{ and } b \in B\}$$

Illustration 1. Suppose $A = \{2, 4, 6\}$ and $B = \{x, y\}$. Then

$$A \times B = \{(2, x), (4, x), (6, x), (2, y), (4, y), (6, y)\}$$
$$B \times A = \{(x, 2), (x, 4), (x, 6), (y, 2), (y, 4), (y, 6)\}$$

Thus, we note that if $A \neq B$, then $A \times B \neq B \times A$.

Illustration 2. Let $A = \{1, 2, 3,\}$ and $B = \phi$. Then

$$A \times B = \phi,$$

because there will be no ordered pair belonging to $A \times B$. Thus, we note that

$$A \times B = \phi$$

if A or B or both of A and B are empty sets.

Illustration 3. Let $n(A)$ represent the number of elements in set A. In Illustration 1 we can see that $n(A) = 3$, $n(B) = 2$ and $n(A \times B) = 6$. Thus, we note that

$$n(A \times B) = n(A) \times n(B)$$

In other words, if a set A has m elements and a set B has n elements, then $A \times B$ has mn elements. Further it may be noted that $n(A \times B) = n(B \times A)$. This implies that $A \times B$ and $B \times A$ are equivalent sets.

Illustration 4. If there are three sets A, B, C and $a \in A$, $b \in B$, $c \in C$, then we form an *ordered triplet* (a, b, c). The set of all ordered triplets (a, b, c) is called the *cartesian product* of the sets A, B and C. That is,

$$A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$$

In particular, $A \times A \times A = \{(a, b, c): a, b, c \in A\}$

Some Results on Cartesian Product of sets

1. Let A, B and C be three sets. Then

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$ (b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(c) $A \times (B - C) = (A \times B) - (A \times C)$ (d) $(A - B) \times C = (A \times B) - (B \times C)$

(e) $(A \cap B) \times C = (A \times C) \cap (B \times C)$ (f) $(A \cup B) \times C = (A \times C) \cup (B \times C)$

2. If A, B and C be any sets and $A \subseteq B$, then $A \times C \subseteq B \times C$.

3. If $A \subseteq B$ and $C \subseteq$, D then $A \times C \subseteq B \times D$.

4. If A, B, C and D be any sets, then $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

5. If A and B be any two non-empty sets, then

$$A \times B = B \times A \Leftrightarrow A = B$$

Example 1. *If* $\left(\frac{x}{3} + 1, y - \frac{2}{3}\right) = \left(\frac{5}{3}, \frac{1}{3}\right)$, *find the values of x and y.*

Solution: $\frac{x}{3} + 1 = \frac{5}{3}$ or $\frac{x}{3} = \frac{5}{3} - 1 = \frac{2}{3}$ $\therefore x = 2$ and $y - \frac{2}{3} = \frac{1}{3}$ or $y = \frac{2}{3} + \frac{1}{3} = 1$

$\therefore x = 3$ and $y = 1$

Example 2. *If the set A has 3 elements and the set* $B = \{3, 4, 5\}$, *find the number of elements in* $A \times B$.

Solution: Set A has 3 elements and set B also has 3 elements, therefore, the number of elements in $A \times B = 3 \times 3 = 9$.

Example 3. *Let* $A = \{1, 2\}$ *and* $B = \{2, 3\}$. *(a) Evaluate* $A \times B, B \times A, A \times A, B \times B$. *(b) Is* $A \times B = B \times A$?

Solution: (a) $A \times B = \{1,2\} \times \{2,3\} = \{ (1,2),(1,3),(2,2),(2,3)\}$

$$B \times A = \{2,3\} \times \{1,2\} = \{ (2,1),(2,2),(3,1)(3,2)\}$$

$$A \times A = \{1,2\} \times \{1,2\} = \{ (1,1),(1,2),(2,1),(2,2)\}$$

$$B \times B = \{2,3\} \times \{2,3\} = \{ (2,2)(2,3),(3,2),(3,3)\}$$

(b) From above results, we conclude that $A \times B \neq B \times A$.

Example 4. *If* $A = \{a, b\}$, $B = \{2, 3, 5, 6, 7\}$ *and* $C = \{5, 6, 7, 8, 9,\}$, *find* $A \times (B \cap C)$.

Solution: We have

$$(B \cap C) = \{2,3,5,6,7\} \cap \{5,6,7,8,9\} = \{5,6,7\}$$

$\therefore$ $A \times (B \cap C) = \{a,b\} \times \{5,6,7\} = \{(a,5),(a,6),(a,7),(b,5),(b,6),(b,7)\}$

Example 5. *If* $A = \{a, d\}$, $B = \{b, c, e\}$ *and* $C = \{b, c, f\}$, *then verify that*

(i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$;

(ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$;

(iii) $A \times (B - C) = (A \times B) - (A \times C)$.

Solution: (i) $(B \cup C) = \{b, c, e\} \cup \{b, c, f\} = \{b, c, e, f\}$

$\therefore$ $A \times (B \cup C) = \{a, d\} \times \{b, c, e, f\}$

$= \{(a,b),(a,c),(a,e),(a,f),(d,b),(d,c),(d,e),(d,f)\}$ ...(1)

Also $(A \times B) = \{(a, b), (a, c), (a, e), (d, b), (d, c), (d, e)\}$

and $(A \times C) = \{(a, b), (a, c), (a, f), (d, b), (d, c), (d, f)$

$\therefore$ $(A \times B) \cup (A \times C) = \{(a,b),(a,c),(a,e),(a,f),(d,b),(d,c),(d,e),(d,f)\}$ ...(2)

From (1) and (2), we have

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

(ii) $(B \cap C) = \{b, c, e\} \cap \{b, c, f\} = \{b, c\}$

$\therefore$ $A \times (B \cap C) = \{a, d\} \times \{b, c\}, = \{(a, b), (a, c), (d, b), (d, c)\}$

Also, $(A \times B) \cap (A \times C) = \{(a, b), (a, c), (d, b), (d, c)\}$

$\therefore$ $A \times (B \cap C) = (A \times B) \cap (A \times C)$

(iii) $(B - C) = \{b, c, e\} - \{b, c, f\} = \{e\}$

$\therefore$ $A \times (B - C) = \{(a, e), (d, e)\}$ ...(3)

Also, $(A \times B) - (A \times C) = \{(a, e), (d, e)\}$ ...(4)

Hence, from (3) and (4), we have

$$A \times (B - C) = (A \times B) - (A \times C)$$

Example 6. *If* $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$, $C = \{1, 3, 4,\}$ *and* $D = \{2, 4, 5\}$, *then verify that*

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

Solution: $(A \times B) = \{1, 2, 3\} \times \{2, 3, 4\}$

$= \{ (1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 2), (3, 3), (3, 4)\}$

$(C \times D) = \{1, 3, 4\} \times \{2, 4, 5\}$

$= \{ (1, 2), (1, 4), (1, 5), (3, 2), (3, 4), (3, 5), (4, 2), (4, 4), (4, 5)\}$

$\therefore$ $(A \times B) \cap (C \times D) = \{ (1, 2), (1, 4), (3, 2), (3, 4)\}$

Also $(A \cap C) = \{1, 3\}$ and $(B \cap D) = \{2, 4\}$

Therefore,

$$(A \cap C) \times (B \cap D) = \{ (1, 2), (1, 4), (3, 2), (3, 4)\}$$

Hence $$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

Example 7. *Let A and B be two sets such that A × B consists of 6 elements. If three elements of A × B are (1, 4), (2, 6), (3, 6), find A × B and B × A.*

Solution: Since (1, 4), (2, 6) and (3, 6) are elements of $A \times B$, therefore, by definition of ordered pair, 1, 2, 3 are elements of the set A and 4, 6 are elements of B. It is given that $A \times B$ has 6 (= 3 × 2) elements. So,

$$A = \{1, 2, 3\} \text{ and } B = \{4, 6\}$$

Hence $A \times B = \{1, 2, 3\} \times \{4, 6\} = \{(1, 4) (1, 6), (2, 4), (2, 6), (3, 4), (3, 6)\}$

and $B \times A = \{4, 6\} \times \{1, 2, 3\} = \{ (4, 1), (4, 2), (4, 3), (6, 1), (6, 2) (6, 3)\}$

Example 8. *If $A \times B = \{(a, 1), (b, 3), (a, 3), (b, 1), (a, 2), (b, 2)\}$, find A and B.*

Solution: In the cartesian product $A \times B$ of two sets A and B, the set A is collection of all first elements in ordered pairs in $A \times B$. Thus, for the given

$$A \times B = \{(a, 1), (b, 3), (a, 3), (b, 1) (a, 2), (b, 2)\};$$

set $A = \{a, b\}$ (dropping the repetitions) and set $B = \{1, 2, 3\}$ is the set of all second elements in ordered pairs in $A \times B$.

Example 9. *State whether each of the following statements are true or false. If the statement is false, rewrite the given statement correctly:*

(i) *If $P = \{m, n\}$ and $Q = \{n, m\}$, then $P \times Q = \{(m, n), (n, m)\}$;*

(ii) *If A and B are non-empty sets, then $A \times B$ is a non-empty set of ordered pairs (x, y) such that $x \in B$ and $y \in A$;*

(iii) *If $A = \{1, 2\}$, $B = \{3, 4\}$, then $A \times (B \cap \phi) = \phi$.*

Solution: (i) False, $P \times Q = \{(m, n), (m, m), (n, n), (n, m)\}$

(ii) False, $A \times B$ is a non-empty set of ordered pairs (x, y) such that $x \in A$ and $y \in B$.

(iii) True, $B \cap \phi = \phi$, therefore, $A \times (B \cap \phi) = A \times \phi = \phi$.

Example. 10. *If $A = \{-1, 1\}$, find $A \times A \times A$.*

Solution:
$$A \times A = \{-1, 1\} \times \{-1, 1\}$$
$$= \{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$$
$$A \times A \times A = \{-1, 1\} \times \{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$$
$$= \{(-1, -1, -1), (-1, -1, 1), (-1, 1, -1), (-1, 1, 1), (1, -1, -1), (1, -1, 1), (1, 1, -1), (1, 1, 1)\}$$

EXERCISE 2.1

LEVEL OF DIFFICULTY A

1. Find the values of a and b in the following equal ordered pairs:

   (i) $(a, 3) = (2, b)$; (ii) $(2a - 3, b + 2) = (3a + 1, 2b + 5)$.

2. If $A = \{1, 3, 5\}$ and $B = \{2, 3\}$, then show that $A \times B \neq B \times A$.

3. If $A = \{a, b\}$, $B = \{c, d\}$, $C = \{d, e\}$, then find

   (i) $A \times (B \cup C)$; (ii) $A \times (B \cap C)$.

4. If $A \times B = \{(a, x), (a, y), (b, x), (b, y)\}$, find A and B.

5. If $A = \{1, 2, 3\}$, $B = \{4\}$, $C = \{5\}$, then verify that:
   (i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$;
   (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$;
   (iii) $A \times (B - C) = (A \times B) - (A \times C)$.

6. If $A = \{1, 4\}$, $B = \{2, 3, 6\}$ and $C = \{2, 3, 7\}$, then verify that:
   (i) $A \times (B \cup C) = (A \times B) \cup (A \times C)$;
   (ii) $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

7. If $A = \{2, 3\}$, $B = \{6, 8\}$, $C = \{1, 2\}$ and $D = \{6, 9\}$, then verify that:
$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D).$$

8. A and B are two sets given in such a way that $A \times B$ contains 6 elements. If three elements of $A \times B$ be (1, 3), (2, 5) and (3, 3), find its remaining elements.

9. Let $A = \{1, 2\}$ and $B = \{3, 4\}$. Write $A \times B$. How many subsets will $A \times B$ have? List them.

10. Let A and B be two sets such that $n(A) = 3$ and $n(B) = 2$. If $(x, 1)$, $(y, 2)$, $(z, 1)$ are in $A \times B$, find A and B, where x, y, z are distinct elements.

11. Let $A = \{1, 2\}$, $B = \{1, 2, 3, 4\}$, $C = \{5, 6\}$ and $D = \{5, 6, 7, 8\}$. Verify that
    (i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$;
    (ii) $A \times C$ is a subset of $B \times D$.

12. The cartesian product $A \times A$ has 9 elements and two of them are $(-1, 0)$ and $(0, 1)$. Find the set A and the remaining elements of $A \times A$.

13. If $P = \{1, 2\}$, form the set $P \times P \times P$.

14. If $G = \{7, 8\}$, $A = \{5, 4, 2\}$, find $G \times H$ and $H \times G$.

15. If R is the set of all real numbers, what do the cartesian products $R \times R$ and $R \times R \times R$ represent?

16. If $A \times B = \{(a, 1), (a, 5), (a, 2), (b, 2), (b, 5), (b, 1)\}$, find $B \times A$.

LEVEL OF DIFFICULTY B

17. (i) If A and B be two non-empty sets having n elements in common, prove that $A \times B$ and $B \times A$ have n^2 elements in common.
    (ii) A and B are two sets having 3 elements in common. If $n(A) = 5$, $n(B) = 4$, find $n(A \times B)$ and $n[(A \times B) \cap (B \times A)]$.

18. Let A be a non-empty set such that $A \times B = A \times C$. Show that $B = C$.

Answers

1. (i) $a = 2, b = 3$; (ii) $a = -4, b = -3$.

3. (i) $\{(a, c), (a, d), (a, e), (b, c), (b, d), (b, e)\}$; (ii) $\{(a, d), (b, d)\}$.

4. $A = \{a, b\}$, $B = \{x, y\}$.

8. (1, 5), (2, 3), (3, 5).

9. $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$.
   Number of subsets of $A \times B = 2^4 = 16$.

Subsets of $A \times B$ are ϕ, {(1, 3)}, {(1, 4)}, {(2, 3)}, {(2, 4)}, {(1, 3), (1, 4)},{(1, 3), (2, 3)}, {(1, 3), (2, 4)}, {(1, 4), (2, 3)}, {(1, 4), (2, 4)}, {(2, 3), (2, 4)}, {(1, 4), (2, 3)}, {(1, 3), (1, 4), (2, 4)}, {(1, 4), (2, 3), (2, 4)}, {(1, 3), (2, 3), (2, 4)}, {(1, 3), (1, 4), (2, 3), (2, 4)}.

10. $A = \{x, y, z\}$, $B = \{1, 2\}$.

12. $A = \{-1, 0, 1\}$; $(-1, -1)$, $(-1, 1)$, $(0, -1)$, $(0, 0)$, $(1, -1)$, $(1, 0)$, $(1, 1)$.

13. {(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)}

14. $G \times H$ = {(7, 5), (7, 4), (7, 2), (8, 5), (8, 4), (8, 2)}
$H \times G$ = {(5, 7), (5, 8), (4, 7), (4, 8), (2, 7), (2, 8)

15. $R \times R = \{(x, y) : x, y \in R\}$ which represents the coordinates of all the points in two dimensional space.
$R \times R \times R = \{(x, y, z) : x, y, z \in R\}$ which represents the coordinates of all the points in three-dimensional space.

HINTS AND SOLUTIONS

12. Since $(-1, 0) \in A \times A$ and $(0, 1) \in A \times A$, therefore,
$(-1, 0) \in A \times A \Rightarrow -1, 0 \in A$ and $(0, 1) \in A \times A \Rightarrow 0, 1 \in A$
$\therefore$ $-1, 0, 1 \in A$
It is given that A has exactly three elements. Hence $A = \{-1, 0, 1\}$
Remaining elements of $A \times A$ are:
$(-1, -1), (-1, 1), (0, 0), (0, -1), (1, 1), (1, -1), (1, 0)$.

16. $B \times A$ can be obtained from $A \times B$ by interchanging the entries of each ordered pair in $A \times B$.

17. (i) We have, $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
$\Rightarrow (A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$
Since, $A \cap B$ has n elements, so $(A \cap B) \times (B \cap A)$ has n^2 elements.
But $(A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$
$\therefore$ $(A \times B) \cap (B \times A)$ has n^2 elements.
Thus, $A \times B$ and $B \times A$ have n^2 elements in common.
(ii) $n(A \times B) = n(A) \cdot n(B) = 5 \cdot 4 = 20$.
Using (i), $n[(A \times B) \cap (B \times A)] = 3^2 = 9$.

18. Let b be an arbitrary element of B.
Then, $(a, b) \in A \times B$ for all $a \in A$
$\Rightarrow (a, b) \in A \times C$ for all $a \in A$ $[\because A \times B = A \times C]$
$\Rightarrow b \in C$
Thus, $b \in B \Rightarrow b \in C$
$\therefore B \subset C$ ... (1)
Now, let c be an arbitrary element of C.
Then, $(a, c) \in A \times C$ for all $a \in A$
$\Rightarrow (a, c) \in A \times B$ for all $a \in A$ $[\because A \times B = A \times C]$
$\Rightarrow C \in B$
Thus, $c \in C \Rightarrow c \in B$
$\therefore C \subset B$ ... (2)
From (1) and (2), we get $B = C$.

RELATIONS

The word *relation* in literal sense indicates a family tie between two persons. If we are considering a set of persons, then there are many types of family relationships that may or may not hold between two persons, such as '*father of*', '*mother of*', '*brother of*', etc.

For example, if we have set of students A = {Asha, Ram, Hari} and set of fathers, B = {Subash, Shyam, Krishan}, then there is a relation '*is child of*' between the elements of sets A and B. If we agree to write R for '*is child of*', then the above relations can be written as:

Asha R Subash, Ram R Shyam, Hari R Krishan.

Thus R = {(Asha, Subash), (Ram, Shyam), (Hari, Krishan)}

i.e., $R = \{(x, y) \mid x \in A, y \in B, x\, R\, y\}$

Thus, the relation '*is child of*' from set A to set B gives rise to subset R of $A \times B$, such that $(x, y) \in R$ if and only if $x\, R\, y$.

A visual representation of this relation R (called an *arrow diagram*) is shown in Fig. 2.1.

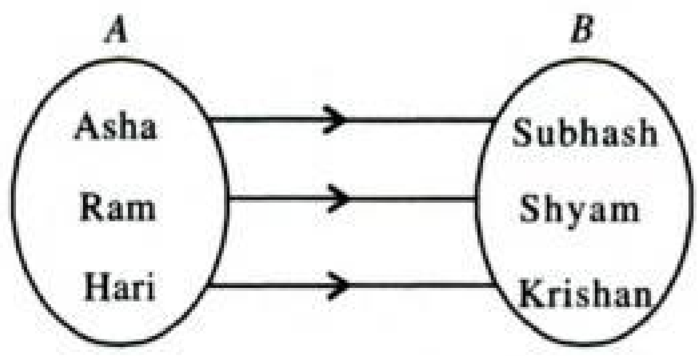

Fig. 2.1

Definition: Let A and B be two non-empty sets. The relation R between A and B is a subset of $A \times B$. The subset is derived by describing a relationship between the first element and the second element of the ordered pairs in $A \times B$. The second element is called the *image* of the first element. Symbolically, we write the relation between A and B as:

$$R : A \to B \text{ if and only if } R \subseteq A \times B$$

IMPORTANT

- A relation may be represented algebraically either by the Roster method or by the set-builder method.
- An arrow diagram is a visual representation of a relation.

Illustration 1. Let I be the set of integers. The statement 'x is less than y for all $x, y \in I$' determines a relation in I. This relation R may be described in the set builder form as:

$$R = \{(1, 2), (2, 3)\ (2, 3), (3, 4), \ldots\}$$
$$= \{(x, y): x, y \in I, x < y\}$$

Illustration 2. Let N be the set of natural numbers. The statement 'x is the cube root of y, for all $x, y \in N$' determines a relation in N. This relation R may be written in the set builder form as:

$$R = \{(1, 1), (2, 8), (3, 27), (4, 64), \ldots\}$$
$$= \{(x, y): x, y \in N \text{ and } y = x^3\}$$

The relation between two elements, say, x and y in a set is also written as $x\, R\, y$ which is read as 'x is R-related to y'.

Domain, Codomain and Range of a Relation

If $R : A \to B$, then the domain of R is the set of all first elements of the ordered pairs (x, y) which belong to R. Symbolically,

$$\text{Domain} = \{x : (x, y) \in R, x \in A \text{ for some } y \in B\}$$

The range of the relation is the set of all second elements of the ordered pairs which belong to R. Symbolically,

$$\text{Range} = \{y : (x, y) \in R, y \in B \text{ for some } x \in A\}$$

The whole set B is called the *codomain* of the relation R. Note that range $\subseteq$ codomain.

Illustration 3. In the illustration 2, if $R \subseteq N \times N$, then the set $\{1, 2, 3, 4, ...\}$ is the domain of R and the set $\{1, 8, 27, 64, ...\}$ is the range of R. Also, the set $\{1, 2, 3, 4, ...\}$ is the codomain of R.

Relation in a Set

Let R be a relation from A to B. If $A = B$, then instead of saying that R is a relation from A to A, we say that R is a relation in A or R is a relation on A.

Thus, a relation R in a set A is a subset of the cartesian product $A \times A$. Symbolically, R is a relation in a set A if and only if $R \subseteq A \times A$.

Inverse Relation

Let R be a relation from A to B. The inverse of R, denoted by R^{-1}, is a relation from B to A and is defined by $R^{-1} = \{(b, a): (a, b) \in R\}$.

Note:
- $(a, b) \in R \Leftrightarrow (b, a) \in R^{-1}$
- Dom (R) = Range (R^{-1}) and Range (R) = Dom (R^{-1})

Illustration 4. Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 4, 5\}$ be two sets and let R be the relation "less than" from A to B.

Then, $R = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$

and $R^{-1} = \{(4, 1), (5, 1), (4, 2), (5, 2), (4, 3), (5, 3), (5, 4)\}$

Also Dom $(R) = \{1, 2, 3, 4\}$ = Range (R^{-1}) and Range $(R) = \{4, 5\}$ = Dom (R^{-1})

Example 1. *Let $A = \{1, 2, 3, ..., 14\}$. Define a relation R from A to A by $R = \{(x, y): 3x - y = 0$, where $x, y \in A\}$. Write down its domain, co-domain and range.*

Solution: $R = \{(x, y): 3x - y = 0, \text{ where } x, y \in A\}$

$= \{(1, 3), (2, 6), (3, 9), (4, 12)\}$

Domain: $\{1, 2, 3, ..., 14\}$

Co-domain: $\{1, 2, 3, ..., 14\}$

Range: $\{3, 6, 9, 12\}$

Example 2. *Let $A = \{1, 2, 3, 4, ... , 250\}$ and R be the relation 'is cube of' in A. Find R as a subset of $A \times A$. Also find the domain and range of R.*

Solution: R is the relation 'is cube of' in A, that is x 'is cube of' y. Therefore,

$$R = \{1, 1), (8, 2), (27, 3), (64, 4), (125, 5), (216, 6)\}$$

Hence, domain of $R = \{1, 8, 27, 64, 125, 216\}$ and range of $R = \{1, 2, 3, 4, 5, 6\}$.

Example 3. *If $A = \{1, 2, 3, 4, 5\}$ and $R = \{(x, y): x > 2, y = 3\}$. Find the domain and range of relation R.*

Solution: Based on the relationship defined among elements of the set R, we can write it as:
$$R = \{(3, 3), (4, 3), (5, 3)\}$$
Domian of R = {3, 4, 5} and range of R = {3, 3, 3} = {3}

Example 4. *If $U = \{1, 2, 3, 4\}$ and $R = \{(x, y) : y > x$ for all $x, y \in U\}$, then find the domain and range of R.*

Solution: The elements of R will be the pairs (x, y) satisfying $y > x$ and $x, y \in U$.

$\therefore$ $$R = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$$
Domain of R = {1, 1, 1, 2, 2, 3}, i.e., {1, 2, 3}

and Range of R = {2, 3, 4, 3, 4, 4, }, i.e., {2, 3, 4}

Example 5. *Find the domain and range of the following relations:*

(i) $R = \{(x, y): x, y \in N, y = x^2 + 3$ *and* $0 < x < 5\}$;

(ii) $R = \{(x, y): (x, y) \in N, y = 1/(1 + x)$ *and x is odd natural number*$\}$.

Solution: (i) We may also write the given relation as:
$$R = \{(x, x^2 + 3): 0 < x < 5, x \in N\}$$
As per condition x takes the values 1, 2, 3, 4 and, therefore, y takes the values 4, 7, 12, 19. Thus,
$$R = \{(1, 4), (2, 7), (3, 12), (4, 19)\}$$
Hence domain of R = {1, 2, 3, 4} and range of R = {4, 7, 12, 19}

(ii) We may also write the given relation as
$$R = \left\{\left(x, \frac{1}{1+x}\right): x \text{ is an odd natural number}\right\}$$
As per condition x takes the values 1, 3, 5, 7, ... Therefore,
$$R = \left\{\left(1, \frac{1}{2}\right), \left(3, \frac{1}{4}\right), \left(5, \frac{1}{6}\right), \left(7, \frac{1}{8}\right), \ldots\right\}$$
Hence domain of R = {1, 3, 5, ...} and range of $R = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots\right\}$

Example 6. *Determine the domain and range of the following relations:*

(i) $R_1 = \left\{\left(x, \frac{1}{x}\right) : 0 < x < 6, x \in N\right\}$;

(ii) $R_2 = \{(x, x^3) : x$ *is prime number less than* $10\}$.

Solution: (i) $R_1 = \left\{\left(x, \frac{1}{x}\right): 0 < x < 6, x \in N\right\} = \left\{\left(x, \frac{1}{x}\right): x = 1, 2, 3, 4, 5\right.$
$$= \left\{(1, 1), \left(2, \frac{1}{2}\right), \left(3, \frac{1}{3}\right), \left(4, \frac{1}{4}\right), \left(5, \frac{1}{5}\right)\right\}$$
Hence domain of R_1 = {1, 2, 3, 4, 5}and range of $R_1 = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$

(ii) $R_2 = \{(x, x^3) : x$ is prime number less than $10\}$
$= \{(x, x^3) : x = 2, 3, 5, 7\} = \{(2, 8,) (3, 27), (5, 125), (7, 343)\}$

Hence domain of R_2 = {2, 3, 5, 7} and range of R_2 = {8, 27, 125, 343}

Example 7. *Define a relation R on the set N of natural numbers by* $R = \{(x, y): y = x + 5, x$ *is a natural number less than* 4; $x, y \in N\}$. *Depict this relationship using* (i) *roster form*; (ii) *an arrow diagram. Write down the domain and the range.*

Solution: We have $R = \{(x, y): y = x + 5, x < 4; x, y \in N\}$

$= \{(1, 6), (2, 7), (3, 8)\}$

Domain = {1, 2, 3}

Range = {6, 7, 8}

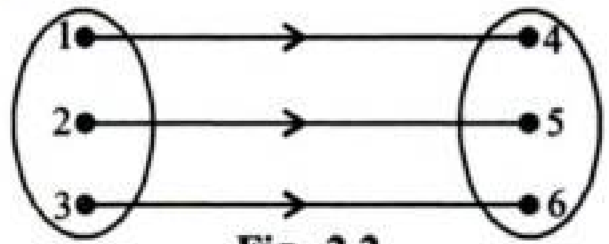

Fig. 2.2

Example 8. $A = \{1, 2, 3, 5\}$ *and* $B = \{4, 6, 9\}$. *Define a relation R from A to B by* $R = \{(x, y)$: *the difference between x and y is odd*; $x \in A, y \in B\}$. *Write R in roster form.*

Solution: We have $R = \{(x, y)$: the difference between x and y is odd, $x \in A, y \in B\}$

$= \{(1, 4), (1, 6), (2, 9), (3, 4), (3, 6), (5, 4), (5, 6)\}$

Example 9. *Let* $A = \{1, 2, 3, 4, 6\}$. *Let R be the relation on A defined by* $\{(a, b): a, b \in A$, *b is exactly divisible by a*$\}$.

(i) *Write R in roster form;*
(ii) *Find the domain of R;*
(iii) *Find the range of R.*

Solution: Clearly, $R = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 6), (2, 2), (2, 4), (2, 6), (3, 3), (3, 6), (4, 4), (6, 6)\}$

(ii) Domain $R = \{1, 2, 3, 4, 6\}$
(iii) Range $R = \{1, 2, 3, 4, 6\}$

Trick(s) for Problem Solving

The total number of relations that can be defined from a set A to a set B is the number of possible subsets of $A \times B$. If $n(A) = p$ and $n(B) = q$, then $n(A \times B) = pq$ and the total number of relations is 2^{pq}.

Example 10. *Let* $A = \{1, 2\}$ *and* $B = \{3, 4\}$. *Find the number of relations from A to B.*

Solution: We have

$$A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$$

Since $n(A \times B) = 4$, the number of subsets of $A \times B$ is 2^4. Therefore, the number of relations from A into B will be 2^4.

Example 11. *Figure 2.3 shows a relationship between the sets P and Q. Write the relation* (i) *in set builder form*; (ii) *roster form. What is its domain and range?*

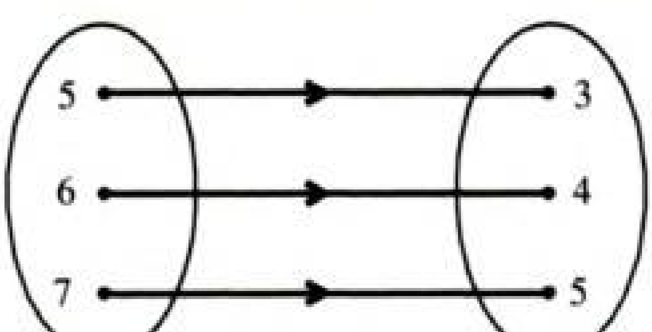

Fig. 2.3

Solution: (i) $R = \{(x, y); x - y = 2, \quad 4 < x < 8, \quad x, y \in N\}$

(ii) $R = \{(5, 3), (6, 4), (7, 5)\}$

Domain = {5, 6, 7}, Range = {3, 4, 5}

Example 12. *A relation R is defined from a set* $A = \{2, 3, 4, 5\}$ *to a set* $B = \{3, 6, 7, 10\}$ *as follows:* $(x, y) \in R \Leftrightarrow x$ *divides y. Express R and* R^{-1} *as a set of ordered pairs. Also, determine the domain and range of R and* R^{-1}.

Solution: Since, 2 divides 6, 2 divides 10, 3 divides 3, 3 divides 6 and 5 divides 10.

$\therefore$ $(2, 6) \in R, (2, 10) \in R, (3, 3) \in R, (3, 6) \in R$ and $(5, 10) \in R$

Thus $R = \{(2, 6), (2, 10), (3, 3), (3, 6), (5, 10)\}$

$\therefore$ $R^{-1} = \{(6, 2), (10, 2), (3, 3), (6, 3), (10, 5)\}$

Also $\text{Dom}(R) = \{2, 3, 5\} = \text{Range}(R^{-1})$

and $\text{Range}(R) = \{3, 6, 10\} = \text{Dom}(R^{-1})$

Example 13. *Let R be the relation on the set Z of all integers defined by* $(x, y) \in R \Rightarrow x - y$ *is divisible by n.*

Prove that:

(i) $(x, x) \in R$ for all $x \in Z$;

(ii) $(x, y) \in R \Rightarrow (y, x) \in R$ for all $x, y \in Z$;

(iii) $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$ for all $x, y, z \in R$.

Solution: (i) For any $x \in Z$,

$x - x = 0 = 0.n$

$\Rightarrow x - x$ is divisible by n

$\Rightarrow (x, x) \in R$

(ii) Let $(x, y) \in R$

$\Rightarrow x - y$ is divisible by n

$\Rightarrow x - y = kn$ for some $k \in Z$

$\Rightarrow y - x = (-k)n$

$\Rightarrow y - x$ is divisible by n

$\therefore$ $(y, x) \in R$.

Thus, $(x, y) \in R \Rightarrow (y, x) \in R$ for all $x, y \in Z$

(iii) Let $(x, y) \in R$ and $(y, z) \in R$

Then $x - y = k_1 n$ and $y - z = k_2 n$ for some $k_1, k_2 \in Z$

$\Rightarrow (x - y) + (y - z) = k_1 n + k_2 n$

$\Rightarrow x - z = (k_1 + k_2)n$

$\Rightarrow x - z$ is divisible by n

$\therefore$ $(x, z) \in R$.

Thus, $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$

EXERCISE 2.2

1. If $A = \{1, 2, 3\}$, then write down the elements of the relation $R = \{(x, y) : x = y\}$ in $A \times A$.
2. If $A = \{4, 5, 9\}$ and $B = \{4, 6, 8\}$, then express a relation R in $A \times B$ if $a < b$, where $a \in A$ and $b \in B$.
3. Find the domain and range of the relation $R = \{(x, y) : x + y = 8$ and x, y are natural numbers$\}$.
4. If $N = \{1, 2, 3\}$, then find the relation $R = \{(x, y) : x \in N, y \in N$ and $2x + y = 10\}$ in $N \times N$.
5. If $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3, 4, 5, 6, 7\}$ and $R = \{(x, y) : x \in A, y \in B$ and $y = x^2 - 3x + 3\}$, write all the elements of R.

6. Let $R = \{(1, 3), (2, 5), (3, 7), (4, 9), (5, 11)\}$ be a relation in the set $A = \{1, 2, 3, \ldots, 11\}$. Find the domain and range of R.

7. Let $A = \{1, 2, 3, \ldots, 70\}$ and R be the relation 'is square of' in A. Find R and its domain and range.

8. Let $A = \{1, 2, 3, \ldots, 30\}$ and R be the relation 'is one-fourth of' in A. Find R. Find also the domain and range of R.

9. Let R be a relation from $A = \{1, 2, 3,\}$ to $B = \{1, 2, 5, 6\}$. Write R as a set of ordered pairs when, (i) R is the relation 'is less than', (ii) R is the relation 'is greater than'.

10. Find the domain and range of the following relations:

(i) $R = \{(x, x^2): x \leq 4, x \in N\}$;

(ii) $R = \{(2x + 3, 1 + x): 3 \leq x \leq 5, x \in N\}$;

(iii) $R = \left\{\left(x+1, \dfrac{1+x^2}{1-x^2}\right): 2 \leq x \leq 4, x \in N\right\}$.

11. Let R be the relation on Z defined by $a\ R\ b$ if and only if $a - b$ is an even integer. Find: (i) R, (ii) domain of R. (iii) range of R.

12. Let R be the relation on Z defined by $R = \{(a, b): a \in Z, b \in Z, a^2 = b^2\}$. Find: (i) R; (ii) domain of R; (iii) range of R.

13. Let R be a relation from Q into Q defined by $R = \{(a, b): a, b \in Q$ and $a - b \in Z\}$. Show that

(i) $(a, a) \in R$ for all $a \in Q$;

(ii) $(a, b) \in R$ implies $(b, a) \in R$;

(iii) $(a, b) \in R$, $(b, c) \in R$ implies $(a, c) \in R$.

14. Let R be a relation from N into N defined by $R = \{(a, b): a, b \in N$ and $a = b^2\}$. Are the following statements true?

(i) $(a, a) \in R$, for all $a \in N$;

(ii) $(a, b) \in R$ implies $(b, a) \in R$;

(iii) $(a, b) \in R$, $(b, c) \in R$ imples $(a, c) \in R$?

Justify your answer in each case.

15. Let $A = \{1, 2, 3, 4\}$ and $B = \{x, y, z\}$. Which of the following are relations from A to B?

(i) $\{(1, x), (2, x), (3, y), (4, z)]$; (ii) $\{(1, 1), (2, x), (3, y), (4, z)]$;

(iii) $\{(1, x), (2, y), (3, z), (4, z), (z, z)\}$; (iv) $A \times B$.

16. Write down the elements of the following relations:

(i) $A = \{(x, y) : x, y \in N \text{ and } x = 3y + 1\}$; (ii) $S = \{(x, y) : x, y \in I \text{ and } x^2 + y^2 \leq 9\}$;

17. Determine the domain and range of the relation R defined by $R = \{(x, x + 5): x \in \{0, 1, 2, 3, 4, 5\}\}$.

18. Let R be the relation on Z defined by $R = \{(a, b): a, b \in Z, a - b$ is an integer$\}$. Find the domain and range of R.

19. Let $A = \{1, 2\}$. List all the relations on A.

20. Let $A = \{x, y, z\}$ and $B = \{1, 2\}$. Find the number of relations from A into B.

21. A relation R is defined on the set Z of integers as follows:

$(x, y) \in R \Leftrightarrow x^2 + y^2 = 25$.

Express R and R^{-1} as the sets of ordered pairs and hence find their respective domains.

22. Let R be the relation on the set N of natural numbers defined by $R = \{(x, y): x + 3y = 12, x \in N, y \in N\}$. Find:

(i) R; (ii) Domain of R; (iii) Range of R.

23. Figure 2.4 given shows a relation R between the sets P and Q. Write this relation R in

(i) Set builder form; (ii) Roster form.

What is its domain and range?

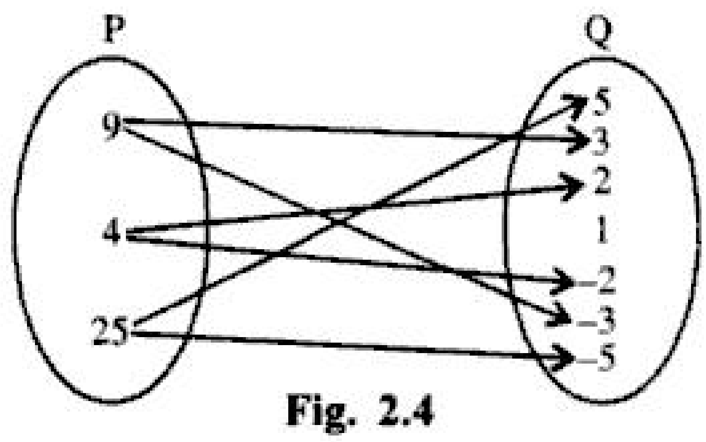

Fig. 2.4

24. Let R be a relation on $N \times N$ defined by

$(a, b)\ R\ (c, d) \Leftrightarrow a + d = b + c$ for all $(a, b), (c, d) \in N \times N$.

Show that:

(i) $(a, b)\ R\ (a, b)$ for all $(a, b) \in N \times N$;

(ii) $(a, b)\ R\ (c, d) \Rightarrow (c, d)\ R\ (a, b)$ for all $(a, b), (c, d) \in N \times N$;

(iii) $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f) \Rightarrow (a, b)\ R\ (e, f)$ for all $(a, b), (c, d), (e, f) \in N \times N$.

25. Let $A = \{7, 11\}$ and $B = \{13, 15\}$. Let $R = \{(a, b): a \in A, b \in B, a - b \text{ is odd}\}$. Show that R is an empty relation from A into B.

Answers

1. (1, 1), (2, 2), (3, 3).

2. $R = \{(4, 6), (5, 6), (5, 8)\}$

3. Domain of $R = \{1, 2, 3, 4, 5, 6, 7\}$; Range of $R = \{7, 6, 5, 4, 3, 2, 1\}$.

4. Domain of $R = \{1, 2, 3, 4\}$; Range of $R = \{8, 6, 4, 2\}$.

5. $\{(1, 1), (2, 1), (3, 3), (4, 7)\}$.

6. Domain of $R = \{1, 2, 3, 4, 5\}$, Range of $R = \{3, 5, 7, 9, 11\}$.

7. $R = \{(1, 1), (4, 2), (9, 3), (16, 4), (25, 5), (36, 6), (49, 7), (64, 8)\}$

Domain of $R = \{1, 4, 9, 16, 25, 36, 49, 64\}$

Range of $R = \{1, 2, 3, 4, 5, 6, 7, 8\}$.

8. $R = \{(1, 4), (2, 8), (3, 12), (4, 16), (5, 20), (6, 24), (7, 28)\}$

Domain of $R = \{1, 2, 3, 4, 5, 6, 7,\}$;

Range of $R = \{4, 8, 12, 16, 20, 24, 28\}$.

9. $A \times B = \{(1, 1), (1, 2), (1, 5), (1, 6), (2, 1), (2, 2) (2, 5), (2, 6), (3, 1), (3, 2), (3, 5), (3, 6)\}$

(i) x 'is less than' y;

$R = \{(1, 2), (1, 5), (1, 6), (2, 5), (2, 6), (3, 5), (3, 6)\}$

(ii) x 'is greater than' y;

$R = \{(2, 1), (3, 1), (3, 2)\}$.

10. (i) Domain $= \{1, 2, 3, 4\}$, Range $= \{1, 4, 9, 16\}$;

(ii) Domain $= \{9, 11, 13\}$, Range $= \{4, 5, 6\}$;

(iii) Domain $= \{3, 4, 5\}$, Range $= \left\{-\frac{5}{3}, -\frac{10}{8}, -\frac{17}{15}\right\}$.

11. (i) $R = \{(a, b): a$ and b are even integers$\} \cup \{(c, d): c$ and d are odd integers$\}$;
(ii) Domain = Z, (iii) Range = Z.

12. (i) $R = \{(a, a): a \in Z\} \cup \{(a, -a): a \in Z\}$;
(ii) Domain = Z; (iii) Range = Z.

14. (i) No; (ii) No; (iii) No.

15. (i) and (iv) are relations from A to B, but (ii) and (iii) are not.

16. (i) $R = \{(4, 1), (7, 2), (10, 3), (13, 4), ...\}$
Domain = $\{4, 7, 10, 13, 16, ...\}$, Range = $\{1, 2, 3, 4, ...\}$;
(ii) $S = \{(0, 0), (1, 1), (-1, 1), (1, -1) (-1, -1), (1, 0), (0, 1), (-1, 0)$
$(0, -1), (2, 2), (-2, 2), (2, -2), (2, 0), (0, 2), (-2, 0), (0, -2), (-2, -2), (3, 0),$
$(-3, 0), (0, -3), (0, 3)\}$
Domain = Range = $\{-3, -2, -1, 0, 1, 2, 3\}$.

17. Domain $(R) = \{0, 1, 2, 3, 4, 5, 6\}$
Range $(R) = \{5, 6, 7, 8, 9, 10\}$.

18. Domain $(R) = Z$, Range $(R) = Z$

19. ϕ, $\{(1, 1)\}$, $\{(1, 2)\}$, $\{(2, 1)\}$, $\{(2, 2)\}$, $\{(1, 1), (1, 2)\}$, $\{(1, 1), (2, 1)\}$, $\{(1, 1) (2, 2)\}$, $\{(1, 2), (2, 1)\}$, $\{(1, 2), (2, 2)\}$, $\{(2, 1), (2, 2)\}$, $\{(1, 1), (1, 2), (2, 1)\}$, $\{(1, 1), (1, 2), (2, 2)\}$, $\{(1, 1), (2, 1), (2, 2)\}$, $\{(1, 2), (2, 1), (2, 2)\}$, $\{(1, 1) (1, 2), (2, 2)\}$

20. 64.

21. $R = \{(0, 5), (0, -5), (3, 4), (-3, 4), (3, -4), (-3, -4), (4, 3), (-4, 3), (4, -3), (-4, -3), (5, 0), (-5, 0)\}$
$R^{-1} = \{(5, 0), (-5, 0) (4, 3), (4, -3), (-4, 3), (-4, -3), (3, 4), (3, -4), (-3, 4), (-3, -4), (0, 5), (0, -5)\}$
Dom $(R) = \{0, 3, -3, 4, -4, 5, -5\}$ = Dom (R^{-1}).

22. (i) $R = \{(9, 1), (6, 2), (3, 3)$; (ii) Dom $(R) = \{9, 6, 3)$; (iii) Range $(R) = \{1, 2, 3)$.

23. (i) $R = \{(x, y): x = y^2, x \in P, y \in Q\}$;
(ii) $R = \{(9, 3), (9, -3), (4, 2), (4, -2), (25, 5), (25, -5)\}$
Dom $(R) = \{9, 4, 25\}$, Range $(R) = \{-5, -3, -2, 2, 3, 5\}$.

HINTS AND SOLUTIONS

13. (i) Since $a - a = 0 \in Z$, it follows that $(a, a) \in R$, for all $a \in Q$;
(ii) $(a, b) \in R$ implies that $a - b \in Z$. So, $b - a \in Z$. Therefore $(b, a) \in R$;
(iii) $(a, b) \in R$, $(b, c) \in R$ implies that $a - b \in Z$, $b - c \in Z$. So, $a - c = (a - b) + (b - c) \in Z$. Hence $(a, c) \in R$.

14. (i) $a = a^2$ is true only when $a = 0$ or 1. Therefore, it is not a relation.
(ii) $a = b^2$ and $b = a^2$ is not true for all $a, b \in N$. Therefore, it is not a relation.
(iii) $a = b^2$, $b = c^2$, $\therefore a = (c^2)^2 = c^4 \Rightarrow a \neq c^2$. Therefore, it is not a relation.

19. Given $A = \{1, 2\}$ $A \times A = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$. Since relation R from set A to set A is a subset of $A \times A$, therefore, all the relations on A are:
Since $n(A \times A) = 4$, the number of all relations in the set $A = 2^4$ or 16.

20. Here $A = \{x, y, z\}$ and $B = \{1, 2\}$. We have $A \times B = \{(x, 1), (x, 2), (y, 1), (y, 2), (z, 1), (z, 2)\}$
Since $n(A \times B) = 6$, the number of subsets of $A \times B$ is $2^6 = 64$. So, the number of relations from A into B is 64.

FUNCTIONS

The concept of a function plays an important role in mathematics. *A* function is a special type of relation as it also helps to indicate the rule of association or correspondence between two elements or objects of two non-empty sets.

Definition: A relation f from a set A to a set B is said to be a function if every element of set A has one and only one image in set B.

In other words, a function f is a relation from a non-empty set A to a non-empty set B such that the domain of f is A and no two distinct ordered pairs in f have the same first element. It is written as:

$$f : A \rightarrow B \text{ such that } x \in A \text{ and } y \in B, \text{ where } y = f(x).$$

If f is a function from A into B, then we write $(x, y) \in f$ as $f(x) = y$, where $x \in A$, $b \in B$. The element $y \in B$ is called the *image* of x under f and x is called the *pre-image* of y under f.

Note that there are many terms such as 'map' or 'mapping' used to denote a function.

Illustration 1. Let $X = \{1, 2, 3\}$, $Y = \{2, 3, 4, 5, 6\}$ and let f be a function defined by

$$f(1) = 2, f(2) = 4, f(3) = 6$$

Figure 2.5 represents two sets and the correspondence between their elements.

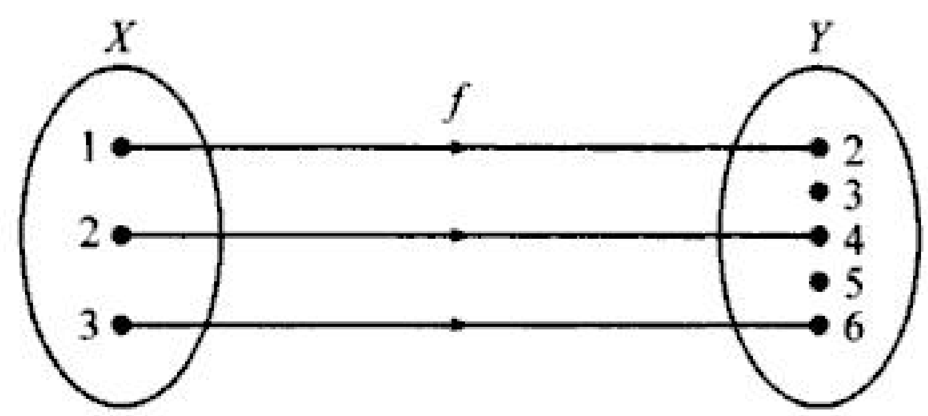

Fig. 2.5

We observe that every element of X has an image in Y (or equivalently, we do not have any element in X which does not have an image in Y) and image of each element of X is unique (or equivalentiy no element of X has more than one image).

Thus, we may say that $f : X \rightarrow Y$ is a function.

Illustration 2. Let $X = \{a, b, c\}$, $Y = \{2, 3, 4, 5\}$ and f be the function defined by

$$f(a) = 2, \quad f(b) = 4$$

Here f is not a function as $c \in X$ does not have its image (see Fig. 2.6).

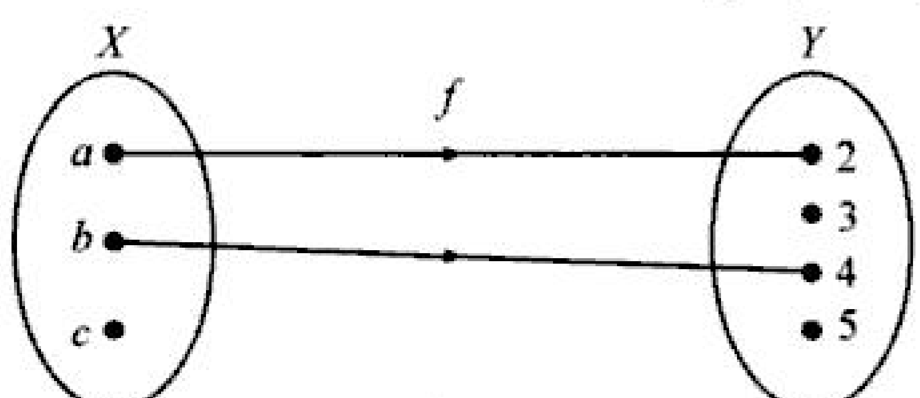

Fig. 2.6

Illustration 3. Let $X = \{a, b, c\}$ and $Y = \{2, 3, 4, 5\}$ and let f be a function defined by

$$f(a) = 2, \ f(b) = 3, \ f(b) = 4, \ f(c) = 5$$

Here every element of X has an image in Y, but there is one element $b \in X$ which does not have unique image. Here b has two images. Hence f is not a function (see Fig. 2.7).

Fig. 2.7

IMPORTANT

In a function $f: X \to Y$,

1. Each element of the set X must be associated with a unique element of Y.
2. Of course, two or more elements of the set X may be associated with the same element of Y.
3. There may be some elements of the set Y which are not assigned to any element of the set X.

Illustration 4. If $f(x) = \dfrac{\sin x}{x}$, then $f(0) = \dfrac{\sin 0}{0} = \dfrac{0}{0}$.

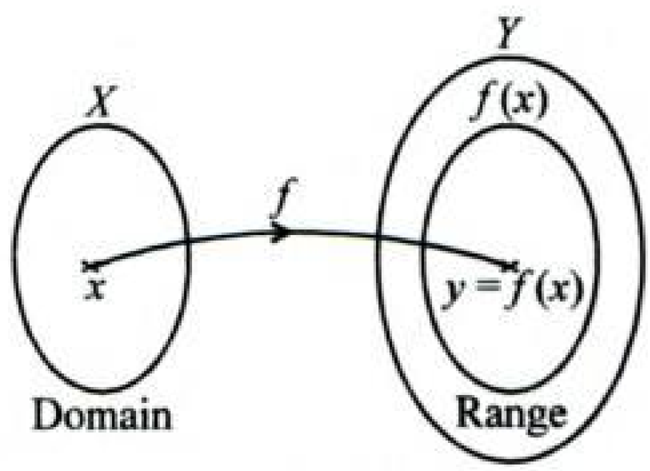

Fig. 2.8

Domain and Range of a Function

If $f: X \to Y$ be a function, then domain of $f = X$,

and range of $f = f(X) = \{f(x) : f(x) \in Y; x \in X\}$

= set of all image points in Y under the map f.

Y is called the *co-domain* of f.

Clearly $f(X) \subseteq Y$.

Illustration 5. If $X = \{0, \pm 1, \pm 2\}$, $Y = \{0, 1, 2, 4\}$, then the rule $f: X \to Y$ given by

$f(x) = x^2$ is a map from X to Y.

$\therefore$ Domain of $f = X$, Range of $f = \{0, 1, 4\}$

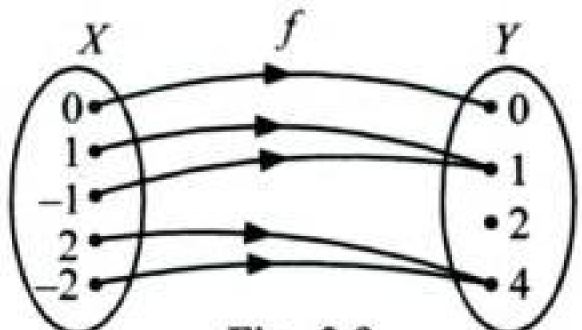

Fig. 2.9

Illustration 6. Let $A = \{-3, -2, -1, 4\}$ and $f: X \to Z$ given by $f(x) = x^2 + x + 2$. Find

(a) the range of f; (b) Pre-images of 6 and 4.

Solution: (a) We have, $f(x) = x^2 + x + 2$

$\Rightarrow f(-3) = (-3)^2 + (-3) + 2 = 8,\quad f(-2) = (-2)^2 + (-2) + 2 = 4$

$f(-1) = (-1)^2 + (-1) + 2 = 2$ and $f(4) = (4)^2 + 4 + 2 = 22$

Therefore, range $(f) = \{8, 4, 2, 22\}$

(b) Let x be the pre-image of 6.

Then, $f(x) = 6 \Rightarrow x^2 + x + 2 = 6 \Rightarrow x^2 + x - 4 = 0 \Rightarrow x = \dfrac{1 \pm \sqrt{17}}{2}$

Since $\dfrac{1 \pm \sqrt{17}}{2} \notin A$, therefore, there is no pre-image of 6.

Let x be the pre-image of 4.

Then $f(x) = 4 \Rightarrow x^2 + x + 2 = 4 \Rightarrow x^2 + x - 2 = 0 \Rightarrow x = -2, 1$

Since $-2 \in A$, therefore, -2 is the pre-image of 4.

Differences between Relation and Function

1. If R is a relation from A to B, then domain of R may be a subset of A. But if f is a function from A to B, then domain f is equal to A.
2. In a relation from A to B, an element of A may be related to more than one element in B. But in a function from A to B, each element of A must be associated to one and only one element of B.

Thus, every function is a relation, but every relation is not necessarily a function.

Illustration 7. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. Let $R = \{(1, a), (2, a), (3, b), (4, b)\}$. Then R is a function from A to B. Obviously, R is also a relation from A to B because $R \subseteq A \times B$. But consider the subset S of $A \times B$ given by $S = \{(1, a), (2, b), (1, c), (4, b)\}$. Here, S is a relation from A to B because $S \subseteq A \times B$. But S is not a function from A to B. The obvious reason is that the element $1 \in A$ is associated to two different elements a and $c \in B$.

GRAPH OF A FUNCTION

The graph of a function $f : A \to B$ is the set of points $(a, f(a))$ in $A \times B$, where $a \in A$.

SOME IMPORTANT FUNCTIONS

Constant Function

A function $f : R \to R$ defined by $f(x) = c$, $\forall\, x \in R$, where c is a constant, is called a *constant function*. Its domain is R and range is singleton set $\{c\}$.

The graph of a constant function is a straight line parallel to x-axis when x is the independent variable (see Fig. 2.10).

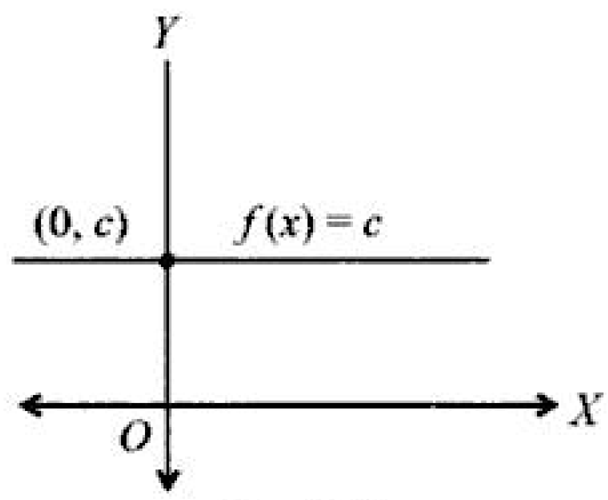

Fig. 2.10

Identity Function

The function $f : R \to R$ defined by $f(x) = x$, $\forall\, x \in R$, is called the *identity function*. Its domain is R and range is also R.

The graph of the identity function is a straight line passing through origin and inclined at an angle of 45° with x-axis (Refer Fig. 2.11).

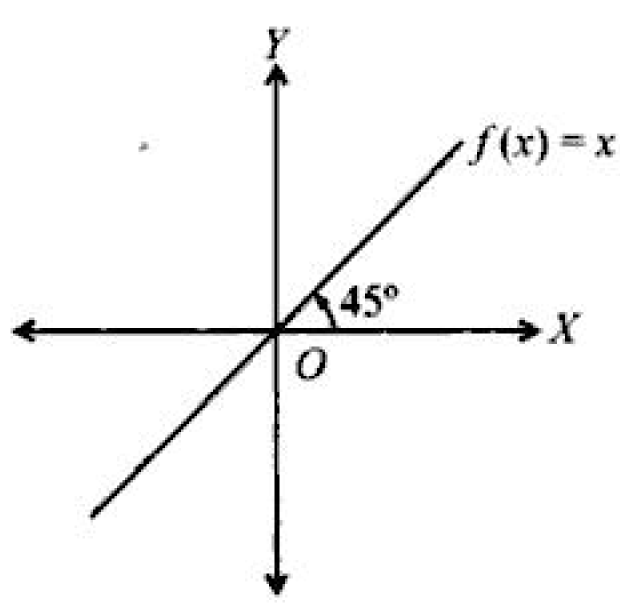

Fig. 2.11

Modulus Function or Absolute Value Function

The function $f : R \to R$, defined by

$$f(x) = |x| = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -x, & \text{if } x < 0 \end{cases}$$

is called the *absolute value function* or *modulus function*. Its domain is R and its range is $(0, \infty)$. The graph of the modulus function is shown in Fig. 2.12.

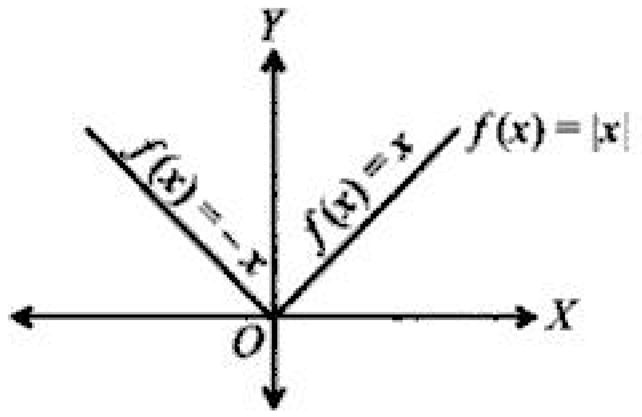

Fig. 2.12

Greatest Integer Function or Step Function or Integral Function

The function $f : R \to R$ defined by $f(x) = [x]$ is called the *greatest integer function*, where $[x]$ = integral part of x or greatest integer not greater than x or greatest integer less than or equal to x.

i.e., $f(x) = n$, where $n \leq x < n + 1$, $n \in I$ (the set of integers).

Its domain is R and range is I. The graph of the greatest integer function is shown in Fig. 2.13.

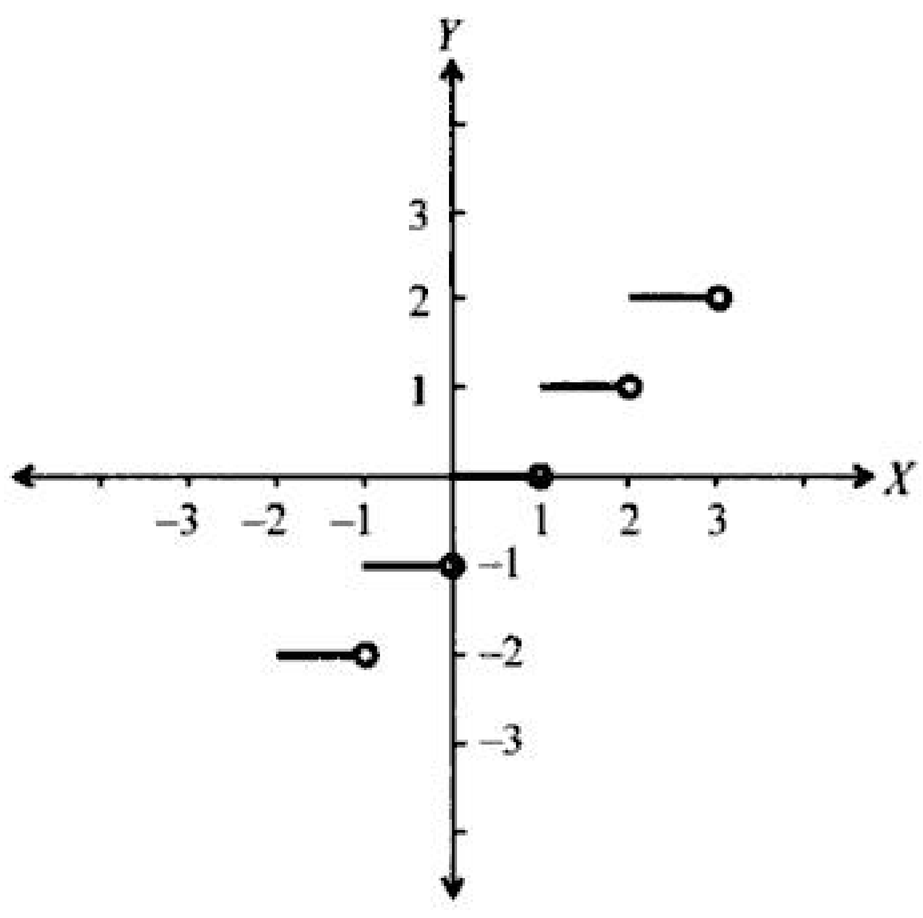

Fig. 2.13

Note:

(a) $[x] \leq x < [x] + 1$

(b) $[x + y] = \begin{cases} (x) + (y), & \text{if } (x) + (y) < 1 \\ (x) + (y) + 1, & \text{if } (x) + (y) \geq 1 \end{cases}$, where $\{x\}$ denotes the fractional part of x.

Fractional-part Function

The function $f : R \to R$ defined by $f(x) = x - [x]$ or $f(x) = \{x\}$, where $\{x\}$ denotes the fractional part of x, is called the *fractional-part function*. Its domain is R and range is $[0, 1)$. The graph of the fractional part function is given in Fig. 2.14.

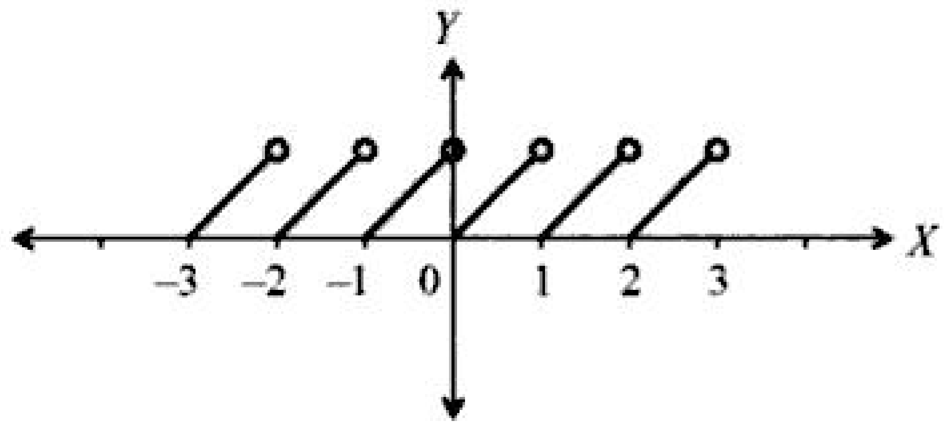

Fig. 2.14

Signum Function

The function $f : R \to R$, defined by

$$f(x) = \left\{ \begin{matrix} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{matrix} \right\}$$, is called the signum function.

Its domain is R and range is the set $\{-1, 0, 1\}$. The graph of the signum function is shown in Fig. 2.15.

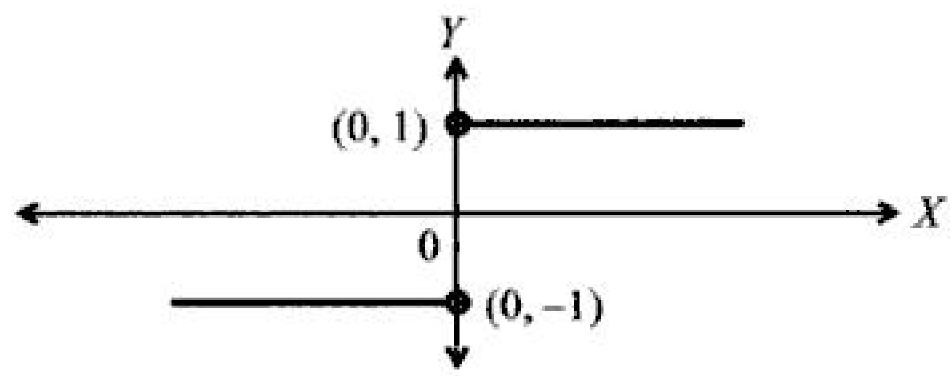

Fig. 2.15

Reciprocal Function

The function $f : R - \{0\} \to R$ defined by $f(x) = \frac{1}{x}$, is called the *reciprocal function*. Its domain as well as range is $R - \{0\}$. The graph of the reciprocal function is exhibited in Fig. 2.16.

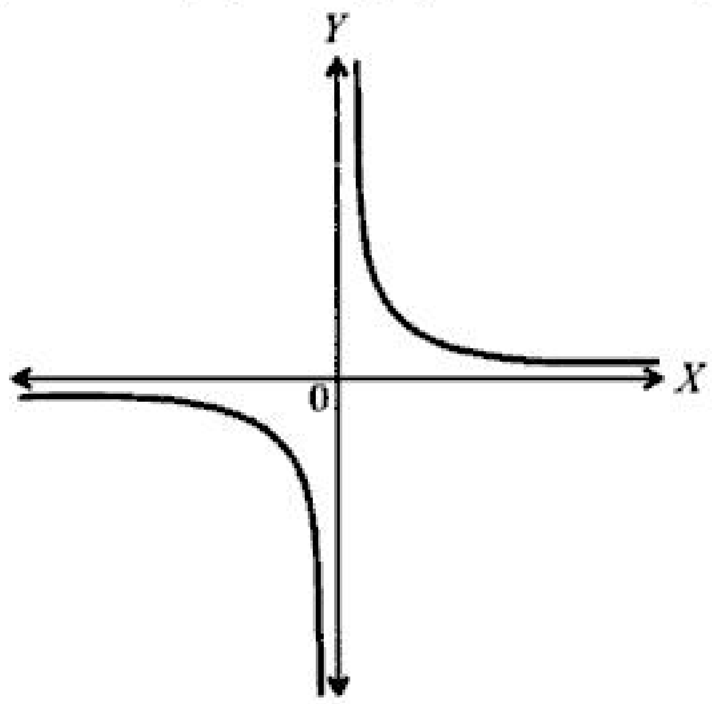

Fig. 2.16

Exponential Function

Let a ($\neq 1$) be a positive real number. Then the function $f : R \to R$, defined by $f(x) = a^x$, is called the *exponential function*. Its domain is R and range is $(0, \infty)$. The graph of the exponential function is given in Fig. 2.17.

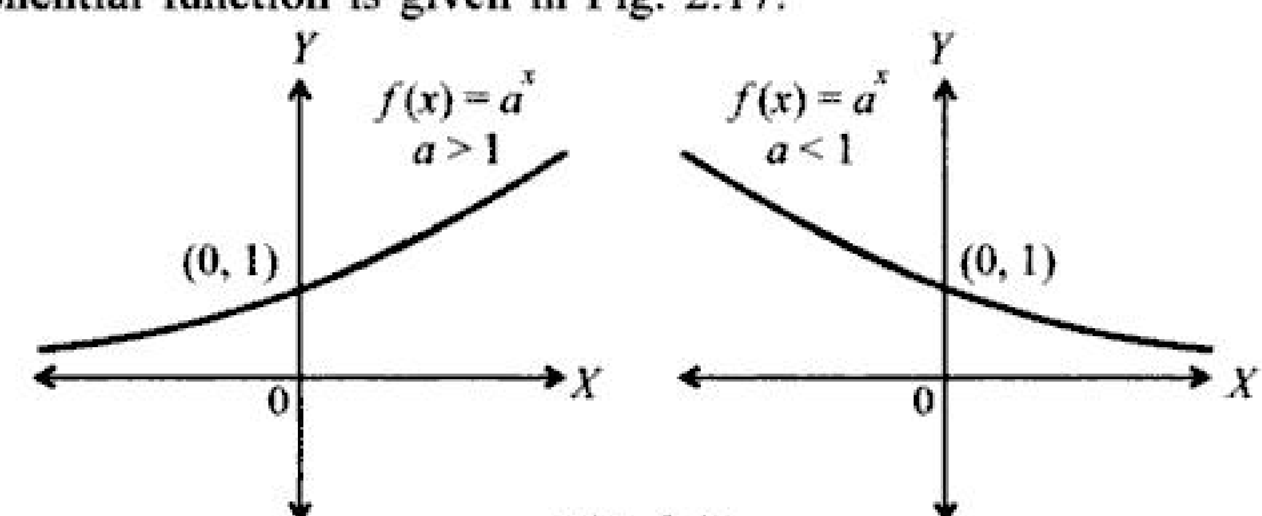

Fig. 2.17

Note: (i) $a^x =$, $(a > 0)$, (ii) $= x$, $(a > 0, a \neq 1)$.

Logarithmic Function

Let $a(\neq 1)$ be a positive real number. Then the function $f: (0, \infty) \to R$, defined by $f(x) = \log_a x$, is called the *logarithmic function*. Its domain is $(0, \infty)$ and range is R. The graph of the logarithmic function is shown in Fig. 2.18.

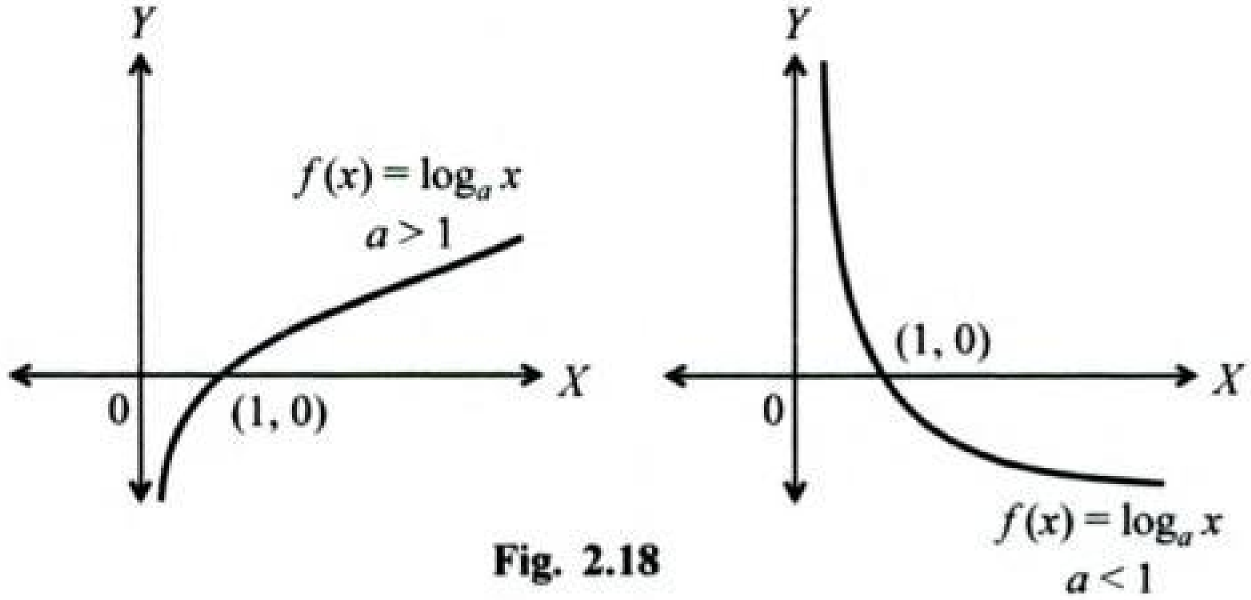

Fig. 2.18

> **IMPORTANT**
>
> (i) $\log_a a = 1$, $\log_a 1 = 0$
>
> (ii) $\log_a 0 = -\infty$, if $a > 1$
>
> $= +\infty$, if $0 < a < 1$
>
> (iii) We denote $\log_e x$ as $\ln x$.

Polynomial Function

A function $f: R \to R$, defined by $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, where $n \in N$, $a_0, a_1, a_2, \ldots, a_n \in R$, is called a *polynomial function*.

If $a_n \neq 0$, then n is called the *degree of the polynomial*. The domain of a polynomial function is R.

Rational Function

A function of the form $f(x) = \dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials over the set of real numbers and $q(x) = 0$, is called a *rational function*. Its domain is $R - \{x \mid q(x) = 0\}$.

TWO WAYS OF DEFINING A FUNCTION

Uniform Definition

If a function is defined as

$$y = f(x),\ x \in [a, b]$$

we say that it is uniformly defined.

Illustration 8. (i) $y = f(x) = \sin x$, $x \in R$; (ii) $y = f(x) = x^2 + 1$, $x \in [-1, 1]$.

Piecewise Definition

If a function $y = f(x)$, $x \in [a, b]$ assumes different forms in different subsets of $[a, b]$, we say that it is piecewise defined.

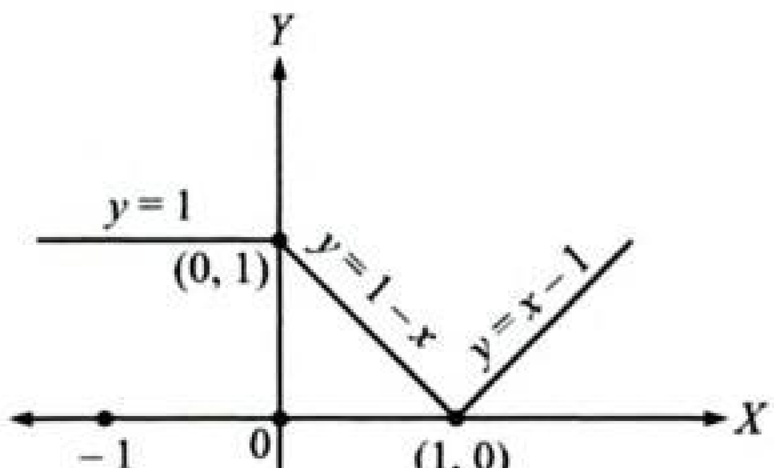

Fig. 2.19

Illustration 9. $y = f(x) = \begin{cases} 1, & -1 \le x < 0 \\ 1 - x, & 0 \le x < 1 \\ x - 1, & x \ge 1 \end{cases}$

EXPLICIT AND IMPLICIT FUNCTIONS

Explicit Function

A function y is said to be an explicit function of x, if the dependent variable y can be expressed totally in terms of the independent variable x.

EQUAL FUNCTIONS

If f and g are functions defined on the same domain A and if $f(a) = g(a)$ for every $a \in A$, then $f = g$.

Illustration 10. Let f: $\{1, 2\} \to \{1, 2, 3, 4\}$ such that $f = \{(1, 1), (2, 3)\}$ and $g = \{1, 2\} \to \{1, 2, 3, 4, 5\}$ such that $g = \{(1, 1), (2, 3)\}$.

Since domain of f = domain of g and $f(a) = g(a)\ \forall\ a$ in the domain,

$$\therefore \quad f = g$$

OPERATIONS ON FUNCTIONS

Let f and g be real functions with domain D_1 and D_2 respectively. Then,

(i) The sum function $(f + g)$ is defined by

$$(f + g)(x) = f(x) + g(x),\ \forall\ x \in D_1 \cap D_2$$

The domain of $f + g$ is $D_1 \cap D_2$.

(ii) The difference function $(f - g)$ is defined by

$$(f - g)(x) = f(x) - g(x),\ \forall\ x \in D_1 \cap D_2$$

The domain of $f - g$ is $D_1 \cap D_2$.

(iii) The product function fg is defined by

$$(fg)(x) = f(x) \cdot g(x),\ \forall\ x \in D_1 \cap D_2$$

The domain of fg is $D_1 \cap D_2$.

(iv) The quotient function is defined by

$$\frac{f}{g}(x) = \frac{f(x)}{g(x)},\ \forall\ x \in D_1 \cap D_2 \backslash \{x : g(x) = 0\}$$

The domain of is $D_1 \cap D_2 \backslash \{x : g(x) = 0\}$.

(v) The scalar multiple function cf is defined by

$$(cf)(x) = c \cdot f(x),\ \forall\ x \in D_1$$

The domain of cf is D_1.

Working Rule for Finding the Domain of a Function

In order to find domain of a function $y = f(x)$, find all those real values of x for which y is defined, i.e., y is real.

For algebraic functions,

(i) Denominator should be non-zero.

(ii) Expression under the even root should be non-negative.

Working Rule for Finding the Range of a Function

Step 1. Put $y = f(x)$.

Step 2. Solve the equation $y = f(x)$ and find x in terms of y to get $x = g(y)$.

Step 3. Find the real values of y for which x is real and is in the domain of f.

Step 4. The set of values of y obtained in Step 3 is the range of f.

Example 1. *Which of the following are functions if* $X = \{a, b, c, d\}$, $Y = \{1, 2, 3, 4, 5\}$.

(i) $f_1 = \{(a, 1), (b, 1), (c, 3), (d, 4)\}$;

(ii) $f_2 = \{(a, 1), (b, 2), (c, 4), (a, 2), (d, 5\}$;

(iii) $f_3 = \{(a, 2), (b, 1), (c, 4), (d, 5)\}$.

Solution: Here $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3, 4, 5\}$.

(i) f_1 is a function because each element of X has unique image, *viz.*

$$f(a) = 1, f(b) = 1, f(c) = 3, f(d) = 4$$

(ii) f_2 is not a function as $a \in X$ has two images, *viz.*

$$f(a) = 1 \text{ and } f(a) = 2$$

(iii) f_3 is a function because each element of X has unique image, *viz.*

$$f(a) = 2, f(b) = 1, f(c) = 4, f(d) = 5.$$

Example 2. *Examine the relation and state whether it is a function or not.*

(i) $y = \pm\sqrt{1 - x^2}$, *for* $x \le 1$; (ii) $y = \sqrt{x}$ *for* $x \ge 0$;

(iii) $y = -(x)$ *for* $x \in R$; (iv) $y = \dfrac{1}{x-1}$ *for* $x \ge 0$.

Solution: (i) Since $\pm\sqrt{1 - x^2}$ is not defined for $x < -1$. Thus, the points $x = -2, -3$, ... , for the domain have no image. Therefore, It is not a function.

(ii) Yes, because for every value of x, we can find a corresponding y.

(iii) Yes, because for every value of x, we can find a corresponding y.

(iv) No, since $f(1)$ does not exist, whereas $1 \in D$.

Example 3. *Let* $X = \{1, 2, 3, 4, 5\}$, $Y = \{1, 2, 5, 6, 7, 9, 10, 11, 12, 13, 14\}$. *Find the function defined by* $f(x) = 2x + 3$. *Also find the domain and range.*

Solution: Here the given rule is $y = f(x) = 2x + 3$.

$\therefore$ $f(1) = 5, f(2) = 7, f(3) = 9, f(4) = 11, f(5) = 13$

$\therefore$ $f = \{(1, 5), (2, 7), (3, 9), (4, 11), (5, 13)\}$

$\therefore$ Domain $= \{1, 2, 3, 4, 5\}$ and Range $= \{5, 7, 9, 11, 13\}$

Example 4. *Write the domain of* $y = \dfrac{x^2 - 9}{x - 3}$.

Solution: Domain is the set of all reals except 3 because at $x = 3$, y is not defined.

$\therefore$ Domain $= R - \{3\}$

Example 5. *Write the range of* $y = \dfrac{|x - 1|}{x - 1}$.

Solution: The value of y will be 1 if $x - 1 > 0$ and -1 if $x - 1 < 0$. Hence the range is $\{-1, 1\}$.

Example 6. *Find the domain and range of the following functions:*

(i) $y = \dfrac{x^2 - 1}{x - 1}, x \neq 1$; (ii) $y = \sqrt{x}, x \geq 0$; (iii) $y = x^3$; (iv) $y = |x|$.

Solution: (i) Since y is defined for all real x except $x = 1$.

$\therefore$ Domain $= R - \{1\}$

For $x \neq 1$, $y = x + 1 \neq 2$

For other real values of x, y is always real. Thus, the range of the function $= R - \{2\}$.

(ii) $y = \sqrt{x}$ is defined only for posititve values of x.

$\therefore$ Domain $=$ All real $x \geq 0$

Also range is all real $x \geq 0$. [$\because$ for $x \geq 0, y \geq 0$]

(iii) $y = x^3$, is defined for all real values of x

$\therefore$ Domain $=$ All real x

Similarly, y can have all real values.

$\therefore$ Range $=$ All real x

(iv) $y = |x|$,

Since x can have all real values.

$\therefore$ Domain $=$ All real x

But y can never be negative.

$\therefore$ Range is all real $x \geq 0$.

Example 7. *Let $A = \{8, 11, 12, 15, 18, 23\}$ and f is a function from $A \rightarrow N$ such that $f(x)$ = highest prime factor of x. Find f and its range.*

Solution: $f(8)$ = highest prime factor of 8 = 2

$f(11)$ = highest prime factor of 11 = 11

$f(12)$ = highest prime factor of 12 = 3

$f(15)$ = highest prime factor of 15 = 5

$f(18)$ = highest prime factor of 18 = 3

$f(23)$ = highest prime factor of 23 = 23

$\therefore$ $f = \{8, 2), (11, 11), (12, 3), (15, 5), (18, 3), (23, 23)\}$

and Range of $= \{2, 3, 5, 11, 23\}$.

Example 8. *Is $g = \{(1, 1), (2, 3), (3, 5), (4, 7\}$ a function? If this is described by the formula $g(x) = \alpha x + \beta$, then what values would be assigned to α and β?*

Solution: Since first entry of each ordered pair has a unique second entry of the ordered pair, i.e., $g(x) = y$ is unique, so g is a function.

$$g(1) = 1 \Rightarrow \alpha + \beta = 1$$

and $$g(2) = 3 \Rightarrow 2\alpha + \beta = 3$$

Solving these equations simultaneously, we get $\alpha = 2$, $\beta = -1$.

Example 9. *Find the domain and range of $f(x) = \sqrt{1 - x^2}$ from R to R.*

Solution: $f(x) = \sqrt{1 - x^2}$

When $1 - x^2 < 0$, $f(x)$ is not defined.

$\therefore$ $f(x)$ is defined when $1 - x^2 \geq 0$

or $-x^2 \geq -1$ or $x^2 \leq 1$

$\therefore$ Domain = $\{x : x \in R, -1 \le x \le 1\}$

For range,

$$y = \sqrt{1-x^2} \quad \text{or} \quad y^2 = 1 - x^2 \quad \therefore \quad x^2 = 1 - y^2$$

or $x = \sqrt{1-y^2}$,

which is defined when

$$1 - y^2 \ge 0 \quad \text{or} \quad -y^2 \ge -1 \quad \text{or} \quad y^2 \le 1, \text{ i.e., } -1 \le y \le 1$$

But $y \ge 0$

$\therefore$ Range = $\{y : y \in R, 0 \le y \le 1\}$.

Example 10. *Let f, g : R → R be defined respectively by*

$f(x) = x + 1$, $g(x) = 2x - 3$. Find $f + g$, $f - g$ and f/g.

Solution:

$$(f+g)(x) = f(x) + g(x) = x + 1 + 2x - 3 = 3x - 2$$

$$(f-g)(x) = f(x) - g(x) = (x+1) - (2x-3) = -x + 4$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = \frac{x+1}{2x-3}, \quad x \in R \text{ and } x \ne \frac{3}{2}.$$

Example 11. *The function 't' which maps temperature in celsius into temperature in degree Fahrenheit is defined by*

$$t(C) = \frac{9C}{5} + 32$$

Find: (i) $t(0)$; (ii) $t(28)$; (iii) $t(-10)$; (iv) *The value of C, when $t(C) = 212$.*

Solution: We have

$$t(C) = \frac{9C}{5} + 32$$

(i) $t(0) = \dfrac{9(-0)}{5} + 32 = 32$

(ii) $t(28) = \dfrac{9(28)}{5} + 32 = \dfrac{252}{5} + 32 = \dfrac{252+160}{5} = \dfrac{412}{5}$

(iii) $t(-10) = \dfrac{9(-10)}{5} + 32 = -18 + 32 = 14$

(iv) $t(C) = 212 \Rightarrow 212 = \dfrac{9C}{5} + 32$ or $\dfrac{9C}{5} = 212 - 32 = 180$

or $C = \dfrac{180 \times 5}{9} = 100$

Example 12. *Find the domain of the function:*

$$f(x) = \frac{x^2 + 2x + 1}{x^2 - 8x + 12}$$

Solution: We have, $f(x) = \dfrac{x^2 + 2x + 1}{x^2 - 8x + 12} = \dfrac{(x+1)^2}{(x-2)(x-6)}$

The function f is not defined at $x = 2, 6$.

$\therefore$ Domain of $f = \{x : x \in R \text{ and } x \neq 2, x \neq 6\}$

$= R - \{2, 6\}$

Example 13. *Find the range of each of the following functions:*

(i) $f(x) = 2 - 3x, x \in R, x > 0$;

(ii) $f(x) = x^2 + 2, x \in R$;

(iii) $f(x) = x, x \in R$.

Solution: (i) Let $f(x) = y = 2 - 3x \Rightarrow x = \dfrac{2-y}{3}$

Now $x > 0 \Rightarrow 2 - y > 0$ or $y < 2$

$\therefore$ Range $(f) = \{y : y \in R \text{ and } y < 2\}$

(ii) Let $f(x) = y = x^2 + 2$

$\Rightarrow x^2 = y - 2$ or $x = \sqrt{y-2} \Rightarrow y > 2$

$\therefore$ Range $(f) = \{y : y \in R \text{ and } y > 2\}$

(iii) Let $f(x) = y = x$, x is a real number.

$\therefore$ $x = f(x) = y =$ a real number

$\therefore$ Range $(f) = \{y : y \in R\}$

Example 14. (i) *If* $f(x) = \dfrac{x^2 - x + 1}{x^2 + x + 1}$, *find* $f(1 + h)$;

(ii) *If* $f(x) = \dfrac{\sin x - \cos x}{\sin x + \cos x}$, *find* $f\left(\dfrac{\pi}{3}\right)$; (iii) *If* $f(1 + x) = x^2 + 1$, *find* $f(2 - x)$;

(iv) *If* $f(x) = \dfrac{x+1}{x-1}$, *find* $f(x^2)$, $[f(x)]^2$.

Solution:

(i) $f(1 + h) = \dfrac{(1+h)^2 - (1+h) + 1}{(1+h)^2 + (1+h) + 1} = \dfrac{h^2 + h + 1}{h^2 + 3h + 3}$

(ii) $f\left(\dfrac{\pi}{3}\right) = \dfrac{\sin \pi/3 - \cos \pi/3}{\sin \pi/3 + \cos \pi/3} = \dfrac{\sqrt{3}/2 - 1/2}{\sqrt{3}/2 + 1/2} = \dfrac{\sqrt{3} - 1}{\sqrt{3} + 1} = 2 - \sqrt{3}$

(iii) Since $f(1 + x) = x^2 + 1$, replace x by $1 - x$, we get

$f(1 + 1 - x) = (1 - x)^2 + 1 \quad \therefore \quad f(2 - x) = x^2 - 2x + 2$

(iv) $f(x^2) = \dfrac{x^2 + 1}{x^2 - 1}$ and $[f(x)]^2 = \left(\dfrac{x+1}{x-1}\right)^2$

Example 15. *Find a polynomial function* $f(x)$ *of the second degree when* $f(0) = 5$, $f(-1) = 10, f(1) = 6$.

Solution: Let $f(x) = ax^2 + bx + c$

$\therefore$ $f(0) = a.\,0^2 + b.0 + c = 5 \quad \therefore \quad c = 5$

Also, $f(-1) = a - b + c = 10 \quad \therefore \quad a - b = 5$ ...(1)

$f(1) = a + b + c = 6 \quad \therefore \quad a + b = 1$ ...(2)

Solving (1) and (2), we get

$$a = 3 \text{ and } b = -2; \quad \therefore \quad f(x) = 3x^2 - 2x + 5.$$

Example 16. *If* $f(x) = 1 + x, \quad -1 \le x < 0$

$= x^2 - 1, \quad 0 < x < 2$

$= 2x, \quad x \ge 2$

find $f(3), f(-2), f(0), f(1/2), f(2-h), f(-1+h), f(f(1/2))$, *where* $h > 0$ *is very small.*

Solution: $f(3) = 2(3) = 6$ $\quad [\because \ f(x) = 2x \text{ for } x \ge 2]$,

$f(-2)$ = not defined; $f(0)$ = not defined

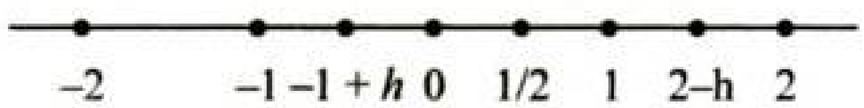

Fig. 2.20

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 - 1 = -\frac{3}{4} \qquad [\because f(x) = x^2 - 1 \text{ for } 0 < x < 2]$$

$$f\left(f\left(\frac{1}{2}\right)\right) = f\left(-\frac{3}{4}\right) = 1 + \left(-\frac{3}{4}\right) = \frac{1}{4}$$

$$f(-1+h) = 1 + (-1+h) = h \qquad [\because f(x) = 1 + x \text{ for } -1 \le x < 0]$$

$$f(2-h) = (2-h)^2 - 1 = h^2 - 4h + 3 \qquad [\because f(x) = x^2 - 1 \text{ for } 0 < x < 2]$$

Example 17. *Let f be defined by* $f(x) = x - 4$ *and g be defined by*

$$g(x) = \begin{cases} \dfrac{x^2 - 16}{x + 4}, & x \ne -4 \\ k, & x = -4 \end{cases}.$$

Find k such that $f(x) = g(x)$ *for all x.*

Solution: We have

$$f(-4) = -4 - 4 = -8 \text{ and } g(-4) = k$$

But $\quad f(x) = g(x)$ for all x.

$\therefore \quad -8 = k$, i.e., $k = -8$

EXERCISE 2.3

LEVEL OF DIFFICULTY A

1. Which of the following relations are functions? Give reasons. If it is a function, determine its domain and range.

   (i) {(2, 1), (5, 1), (8, 1), (11, 1), (14, 1), (17, 1)};

   (ii) {(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6), (14, 7)};

   (iii) {(0, 0), (1, 1), (1, – 1), (4, 2), (4, – 2), (9, 3) (9, – 3), (16, 4), (16, – 4)};

   (iv) {(1, 3), (1, 5), (2, 5)}; (v) {(2, 1), (3, 1), (5, 2)}; (vi) {(1, 2), (2, 2), (3, 2)}.

2. Let $A = \{3, 4, 5, 6\}$ and $B = \{7, 8, 9, 10, 11, 12\}$. Which of the following are functions from A to B?

   (i) the relation $R = \{(3, 8), (3, 10), (4, 11), (5, 10), (6, 12)\}$;

   (ii) the relation $R = \{(3, 8), (5, 10), (6, 12)\}$;

   (iii) $f : A \to B$ defined by $f(3) = 8, f(4) = 9, f(5) = 10, f(6) = 11$;

(iv) $f: A \to B$ defined by $f(3) = 8, f(4) = 8, f(5) = 10, f(6) = 10$;
(v) $f: A \to B$ defined by $f(3) = 8, f(4) = 9, f(3) = 10, f(5) = 11, f(6) = 7$;
(vi) $f: A \to B$ defined by $f(3) = 8, f(4) = 8, f(5) = 8, f(6) = 8$.

3. Find the domain and range of the following functions:

(i) $\left\{\left(x, \frac{x^2-1}{x-1}\right): x \in R, x \neq 1\right\}$; (ii) $\{(x, -|x|): x \in R\}$;

(iii) $\left\{\left(x, \sqrt{9-x^2}\right): x \in R\right\}$; (iv) $\left\{\left(x, \frac{1}{1-x^2}\right): x \in R, x \neq 1\right\}$.

4. Let f be the function defined by the rule $f(x) = 4x^2 + 2x - 3$. Find $f(2), f(-1), f(a)$ and $f\{f(1)\}$.

5. Let f be a relation on the set N of natural numbers defined by $f = \{(n, 3n): n \in N\}$. Is f a function from N to N? If so, find the range of f.

6. Let $A = \{-2, -1, 0, 1, 2\}$ and $f: A \to Z$ given by $f(x) = x^2 - 2x - 3$. Find
(a) the range of f; (b) pre-images of 6, -3 and 5.

7. Let $A = \{p, q, r, s\}$ and $B = \{1, 2, 3\}$. Which of the following relations from A to B is not a function?
(i) $R_1 = \{(p, 1), (q, 2), (r, 1), (s, 2)\}$; (ii) $R_2 = \{(p, 1), (q, 1), (r, 1), (s, 1)\}$;
(ii) $R_3 = \{(p, 1), (q, 2), (p, 2), (s, 3)\}$; (iv) $R_4 = \{(p, 2), (q, 3), (r, 2), (s, 2)\}$.

8. Let $A = \{12, 13, 14, 15, 16, 17\}$ and $f: A \to Z$ be a function given by $f(x)$ = highest prime factor of x. Find range of f.

9. Let $f = \{(1, 1), (2, 3), (0, -1), (-1, -3)\}$ be a function from Z into Z defined by $f(x) = ax + b$, for some integers a and b. Determine a and b.

10. Let $f = \{(1, 1), (2, 3), (3, 5), (4, 7)\}$ be a function from Z into Z defined by $f(x) = ax + b$, for some integers a and b. Determine a and b.

11. Let $A = \{1, 2, 3, 4\}$. $B = \{1, 5, 9, 11, 15, 16\}$ and $f = \{(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)\}$. Are the following true?
(i) f is a relation from A into B;
(ii) f is a function from A into B;
Justify your answer in each case.

12. Let f be the subset of $Q \times Z$ defined by $f = \left\{\left(\frac{m}{n}, m\right): m \in Z, n \in Z, n \neq 0\right\}$. Is f a function from Q into Z? Justify your answer.

13. Let f be a subset of $Z \times Z$ defined by $f = \{(ab, a + b): a, b \in Z\}$. Is f a function from Z into Z? Justify your answer.

14. Let $f(x) = \sqrt{x}$ and $g(x) = x$ be two functions defined over the set of non-negative real numbers. Find $(f + g)(x)$, $(f - g)(x)$, $(fg)(x)$, $\left(\frac{f}{g}\right)(x)$.

15. Find the domain of the function $f(x) = \dfrac{x^2 + 3x + 5}{x^2 - 5x + 4}$.

16. The relation f is defined by $f(x) = \begin{cases} x^2, 0 \le x \le 3 \\ 3x, 3 \le x \le 10 \end{cases}$

The relation g is defined by $g(x) = \begin{cases} x^2, 0 \le x \le 2 \\ 3x, 2 \le x \le 10 \end{cases}$

Show that f is a function and g is not a function.

17. If $f(x) = x^2$, find $\dfrac{f(1.1) - f(1)}{(1.1 - 1)}$.

18. Find the domain and range of the real function f defined by $f(x) = \sqrt{x - 1}$.

19. Find the domain and range of the real function f defined by $f(x) = |x - 1|$.

20. Let $f = \left\{\left(x, \dfrac{x^2}{1 + x^2}\right) : x \in R\right\}$ be a function from R into R. Determine the range of f.

21. Let $A = \{9, 10, 11, 12, 13\}$ and $f : A \to N$ be defined by $f(n)$ = the highest prime factor of n. Find the range of f.

22. Draw the graph of the following functions:
(i) $f : R \to R$ such that $f(x) = 4 - 2x$;
(ii) $f : R \to R$ such that $f(x) = |x - 2|$.

23. Draw the graph of the function:

$$f(x) = \begin{cases} 1 & \text{for} \quad x > 0 \\ 0 & \text{for} \quad x = 0, x \in R \\ -1 & \text{for} \quad x < 0 \end{cases}$$

24. A function f is defined by $f(x) = 2x - 5$. Write down the values of (i) $f(0)$; (ii) $f(7)$; (iii) $f(-3)$.

25. If $f(x) = \dfrac{5x^2 + 1}{2 - x}$, find $f(3x)$, $f(x^3)$, $3f(x)$, $[f(x)]^3$.

26. Find a linear function $f(x)$ if $f(0) = 2$ and $f(1) = -1$.

LEVEL OF DIFFICULTY B

27. If $f(xy) = f(x) \cdot f(y)$ for all x, y, find $f(1)$ if $f(0) \ne 0$.

28. If $f(x) = \begin{cases} 2 + 3x, & -1 \le x < 1 \\ 3 - 2x, & 1 < x \le 2 \end{cases}$

find $f(0)$, $f(1)$, $f(1 + h)$, $f(2 - h)$, $f(2)$, $f(f(1.5))$.

29. If $f(x) = \dfrac{1 + x}{1 - x}$, show that $\dfrac{f(x) \cdot f(x^2)}{1 + [f(x)]^2} = \dfrac{1}{2}$ and $f(f(x)) = -\dfrac{1}{x}$.

30. If $f(x) = \dfrac{b(x - a)}{b - a} + \dfrac{a(x - b)}{a - b}$, show that $f(a + b) = f(a) + f(b)$.

31. If $f(x) = \dfrac{|x|}{x}$, $x \ne 0$, show that $|f(\alpha) - f(-\alpha)| = 2$, where $\alpha \ne 0$.

32. If $y = f(x) = \dfrac{1 - x}{1 + x}$, show that $x = f(y)$.

33. If $y = f(x) = (a - x^n)^{1/n}$, n any positive integer, show that

(a) $x = f(y)$; (b) $x = f(f(x))$.

34. If $f(x) = \frac{1}{2x+1}$, $x \neq -\frac{1}{2}$, then show that $f(f(x)) = \frac{2x+1}{2x+3}$, provided $x \neq -\frac{3}{2}$.

35. Find the domain of the following functions:

(i) $y = \frac{x}{\sqrt{(1-x)(x-2)}}$; (ii) $y = \sqrt{\frac{x-1}{x-3}}$;

(iii) $y = \frac{1}{\sqrt{|x|-x}}$; (iv) $y = \frac{1}{\sqrt{x^2-5x+6}}$.

36. Find the range of the following functions:

(i) $y = \frac{x}{1+x^2}$; (ii) $y = \frac{x^2-3x+2}{x^2+x-6}$;

37. Find $f + g$; $f - g$, $\alpha f(\alpha \in R)$, $f.g$, $\frac{1}{f}$ and $\frac{f}{g}$ if

(i) $f(x) = \sqrt{x-1}$, $g(x) = \sqrt{x+1}$; (ii) $f(x) = \frac{1}{x+4}$, $g(x) = (x+4)^3$;

(iii) $f(x) = x^3 + 1$, $g(x) = x + 1$.

Answers

1. (i) Domain = {2, 5, 8, 11, 14, 17}, Range = {1};
(ii) Domain = {2, 4, 6, 8, 10, 12, 14}, Range = {1, 2, 3, 4, 5, 6, 7};
(iii) No; (iv) No, 1 has appeared more than once as first component of the ordered pairs, therefore, it is not a function;
(v) Domain = {2, 3, 5}, Range = {1, 2};
(vi) Domian = {1, 2, 3}, Range = {2}.

2. (i) The relation R is not a function as $3 \in A$ is the first element of two elements (3, 8), (3, 10) in R;
(ii) The relation R is not a function as $4 \in A$ is not the first element of any element in R;
(iii) f is a function as to each element of A there corresponds exactly one element of B. In this case, the images are distinct.
(iv) f is a function as to each element of A there corresponds exactly one element of B;
(v) f is not a function as there are two elements 8 and 10 of B which correspond to the same element 3 of A;
(vi) f is a function as to each element of A there corresponds exactly one element of B. In this case, the images of all elements of A is the same element 8 of B.

3. (i) Domain = $R - \{1\}$, Range = $R - \{2\}$;
(ii) Domain = R, Range = $\{x : x \in R \text{ and } x \leq 0\}$;
(iii) Domain = $\{x : x \in R \text{ and } -3 \leq x \leq 3\}$, Range = $\{y : y \in R \text{ and } -3 \leq y \leq 3\}$;
(iv) Domain = $R - \{1, -1\}$, Range = $\{y : y \in R, y \neq 0, y < 0 \text{ and } y \geq 1\}$.

4. 17, -1, $4a^2 + 2a - 3$, 39.

5. Yes, Range of $f = \{3n : n \in N\}$.

6. (a) Range of $f = \{0, 5, -3, -4\}$;
(*b*) No pre-image of 6; 0, 2 are pre-images of -3; -2 is pre-image of 5.

7. (*c*). **8.** {3, 13, 7, 5, 2, 17}. **9.** $a = 2$, $b = -1$.

10. $a = 2$, $b = -1$.

11. (i) Yes, as each element of f belongs to $A \times B$;
(ii) No, as $2 \in A$ has two images viz. 9 and 11 in B.

12. No, $\frac{2}{1} \in Q$ and $\frac{4}{2} \in Q$

$\Rightarrow \left(\frac{2}{1}, 2\right) \in f$ and $\left(\frac{4}{2}, 4\right) \in f$

$\Rightarrow (2, 2) \in f$ and $(2, 4) \in f$ so f is not a function from Q to Z. **13.** No.

14. $(f + g)(x) = \sqrt{x} + x$, $(f - g)(x) = \sqrt{x} - x$, $fg(x) = x^{3/2}$, $\left(\frac{f}{g}\right)(x) = x^{-\frac{1}{2}}$, $x \neq 0$.

15. $R - \{1, 4\}$. **17.** 2.1.

18. Domain $(f) = \{x : x \geq 1\}$
Range $(f) = \{y : y \in R, y \geq 0\}$.

19. Domain $(f) = R$, Range $(f) = \{y : y \geq 0\}$

20. $\{y : y \in R \text{ and } y \in (0, 1)\}$. **21.** $\{3, 5, 11, 13\}$. **24.** (i) 9; (ii) 9; (iii) -11.

25. $\frac{45x^2 + 1}{2 - 3x}, \frac{5x^6 + 1}{2 - x^3}, \frac{15x^2 + 3}{2 - x}, \left(\frac{5x^2 + 1}{2 - x}\right)^3$. **26.** $f(x) = -3x + 2$. **27.** 1.

28. 2, not defined, $1 - 2h$, $2h - 1$, -1, 2.

35. (i) (1, 2); (ii) $[-\infty\ 1] \cup (3, \infty)$; (iii) $(-\infty, 0)$; (iv) $(-\infty, 2) \cup (3, \infty)$.

36. (i) $\left[-\frac{1}{2}, \frac{1}{2}\right]$; (ii) $(-\infty, \infty)$.

37. (i) $\sqrt{x-1} + \sqrt{x+1}, \sqrt{x-1} - \sqrt{x+1}, \alpha\sqrt{x-1}\ \sqrt{x^2-1}, \frac{1}{\sqrt{x-1}}, x \geq 1, \frac{\sqrt{x-1}}{\sqrt{x+1}}, x \geq -1;$

(ii) $\frac{1 + (x+4)^4}{x+4}, \frac{1 - (x+4)^4}{x+4}, \frac{\alpha}{x+4}, (x + 4)^2, x + 4, \frac{1}{(x+4)^4}, x \neq -4$

(iii) $x^3 + x + 2,\ x^3 - x,\ \alpha x^3 + \alpha,\ x^4 + x^3 + x + 1,\ \frac{1}{x^3+1}, x \neq -1, \frac{x^3+1}{x+1}, x \neq -1.$

HINTS AND SOLUTIONS

3. (i) Let $f = \left\{\left(x, \frac{x^2 - 1}{x - 1}\right) : x \in R, x \neq 1\right\}$

Clearly, f is not defined when $x = 1$

$\therefore$ f is defined for all real values of x, except $x = 1$

$\therefore$ Domain $(f) = R - \{1\}$

Let $y = \frac{x^2 - 1}{x - 1} = x + 1$ (as $x \neq 1$) $\Rightarrow x = y - 1$

Clearly, x is not defined, when $y = 2$ as $x \neq 1$

$\therefore$ Range $(f) = R - \{2\}$.

(ii) Let $f = \{(x, -|x|) : x \in R\}$

Clearly, $f(x) = -|x|$ is ≤ 0, for all $x \in R$

$\therefore$ Domain $(f) = R$

and Range $(f) = \{x \in R : x \leq 0\}$

(iii) Let $f = \left\{\left(x, \sqrt{9-x^2}\right) : x \in R\right\}$

Clearly, f is not defined when $(9 - x^2) < 0$, i.e., when $x^2 > 9 \Rightarrow$ When $x > 3$ or $x < -3$.

So domain $(f) = \{x : x \in R : -3 \leq x \leq 3\}$

Further $y = \sqrt{(9-x^2)} \Rightarrow y^2 = 9 - x^2 \Rightarrow x = \sqrt{(9-y^2)}$

Clearly, x is not defined when $(9 - y^2) < 0$

But $(9 - y^2) < 0 \Rightarrow y^2 > 9 \Rightarrow y > 3$ or $y < -3$

$\therefore$ Range $(f) = \{y \in R : -3 \leq y \leq 3\}$

(iv) Let $f = \left\{\left(x, \dfrac{1}{1-x^2}\right) : x \in R\ x \neq \pm 1\right\}$

Clearly, $f(x) = \dfrac{1}{1-x^2}$ is not defined when $(1 - x^2) = 0$, i.e., when $x = \pm 1$

$\therefore$ Domain $(f) = R - \{1, -1\}$

Further $y = \dfrac{1}{1-x^2} \Rightarrow (1 - x^2) = \dfrac{1}{y} \Rightarrow x = \sqrt{\left(1 - \dfrac{1}{y}\right)}$

Clearly, x is not defined, when $\left(1 - \dfrac{1}{y}\right) < 0$ or $1 < \dfrac{1}{y}$ or $y < 1$

$\therefore$ Range $(f) = R - \{y \in R : y < 1\} = \{y \in R : y \geq 1\}$

5. Since for each $n \in N$, there exists a unique $3n \in N$ such that $(n, 3n) \in f$.
Therefore, f is a function from N to N.
Now, Range of $f = \{f(n) : n \in N\} = \{3n : n \in N\}$

6. (*a*) We have:

$$f(-2) = (-2)^2 - 2(-2) - 3 = 5, f(-1) = (-1)^2 - 2(-1) - 3 = 0,$$
$$f(0) = -3, f(1) = 1^2 - 2 \times 1 - 3 = -4 \text{ and } f(2) = 2^2 - 2 \times 2 - 3 = -3$$

So Range $(f) = \{0, 5, -3, -4\}$

(*b*) Let x be the pre-image of 6. Then

$$f(x) = 6 \Rightarrow x^2 - 2x - 3 = 6 \Rightarrow x^2 - 2x - 9 = 0 \Rightarrow x = 1 \pm \sqrt{10}.$$

Since $x = 1 \pm \sqrt{10} \notin A$. So, there is no pre-image of 6.
Let x be the pre-image of -3. Then

$$f(x) = -3 \Rightarrow x^2 - 2x - 3 = -3 \Rightarrow x^2 - 2x = 0 \Rightarrow x = 0, 2$$

Clearly, $0, 2 \in A$. So, 0 and 2 are pre-images of -3.
Let x be the pre-image of 5. Then

$$f(x) = 5 \Rightarrow x^2 - 2x - 3 = 5 \Rightarrow x^2 - 2x - 8 = 0$$
$$\Rightarrow (x - 4)(x + 2) = 0 \Rightarrow x = 4, -2$$

Since, $-2 \in A$. So, -2 is the pre-image of 5.

9. Now, $(1, 1) \in f$ implies $f(1) = 1$ and $(2, 3) \in f$ implies $f(2) = 3$. So, $a + b = 1$ and $2a + b = 3$. Therefore, $a = 2$ and $b = -1$. Then, note that $f(0) = -1$, $f(-1) = -3$.

13. We observe that:

$$1 \times 6 = 6 \text{ and } 2 \times 3 = 6 \Rightarrow (1 \times 6,\ 1 + 6) \in f \text{ and } (2 \times 3,\ 2 + 3) \in f$$
$$\Rightarrow (6, 7) \in f \text{ and } (6, 5) \in f$$

So, f is not a function from Z to Z.

16. $f(x) = x^2$ is well defined in the interval $0 \le x \le 3$. Also, $f(x) = 3x$ is well defined in the interval $3 \le x \le 10$.

At $x = 3$, from $f(x) = x^2$, $f(3) = 3^2 = 9$

Also, $f(x) = 3x$, $f(3) = 3 \times 3 = 9$

$\therefore$ f is defined at $x = 3$. Hence f is a function.

Now, $g(x) = x^2$ is well defined in the interval $0 \le x \le 2$. $g(x) = 3x$ is also well defined in the interval $2 \le x \le 10$

But at $x = 2$, $g(x) = x^2 = 2^2 = 4$

Also, $g(x) = 3x = 3 \times 2 = 6$

At $x = 2$, relation on g has two values.

$\therefore$ Relation g is not a function.

19. $f(x) = |x - 1|$ is defined for all real values of x.

$\therefore$ Domain $(f) = R$

$f(x) = |x - 1|$ can acquire only non-negative values.

$\therefore$ Range $(f) = \{ y : y \ge 0\}$

20. Let $y = f(x) = \dfrac{x^2}{1 + x^2}$

$f(x)$ is positive for all values of x. When $x = 0$, $y = 1$, Also, denominator > numerator.

$\therefore$ Range $(f) = \{ y : y \in R \text{ and } y \in (0, 1)\}$

21. The highest prime factors of 9, 10, 11, 12 and 13 are 3, 5, 11, 3 and 13 respectively.

$\therefore$ Range of $f = \{3, 5, 11, 13\}$

22. (i) Let $y = 4 - 2x$. We know that a linear equation in x and y represents a line. For drawing a line, we need only two points. The following points satisfy the given equation.

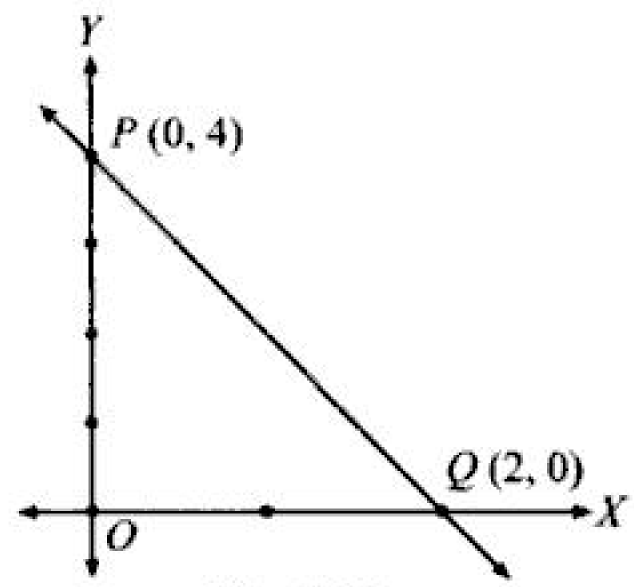

Fig. 2.21

x	0	2
y	4	0

So, plot the point $P(0, 4)$ and $Q(2, 0)$ on a graph paper and join them to get the line PQ, which is the required graph of the given function [see Fig. 2.21]

(ii) Clearly $y = |x - 2| = \begin{cases} x - 2, \text{ for } x - 2 \ge 0, \text{i.e., } x \ge 2 \\ -(x - 2), \text{ for } x - 2 < 0, \text{i.e., } x < 2 \end{cases}$

We know that a linear equation in x and y represents a line. For drawing a line, we need only two points.

For $y = x - 2$

x	2	4
y	0	2

For $y = 2 - x$

x	1	0
y	1	1

So plot the point $P(2, 0)$, $Q(4, 0)$, and join PQ to get the graph of $y = x - 2$. Plot point $R(1, 1)$ and $S(0, 2)$ and join RS to get the graph of $y = 2 - x$. The graph of $y = |x - 2|$ is shown in Fig. 2.22.

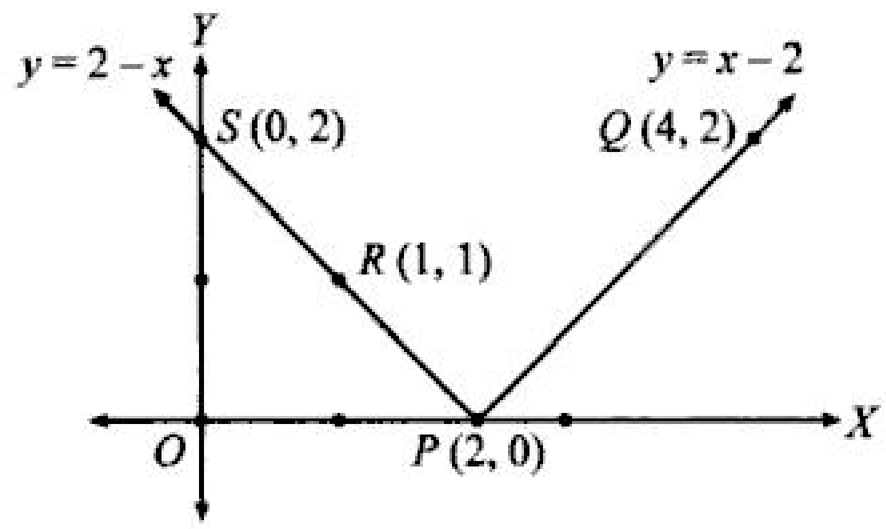

Fig. 2.22

23. We have $f(x) = \begin{cases} 1, & \text{for } x > 0 \\ 0, & \text{for } x = 0, x \in R \\ -1, & \text{for } x < 0 \end{cases}$

Graph of f consists of three parts *viz.*

(i) When $x > 0$, then $y = 1$ is a straight line parallel to x-axis at a distance of 1 unit above x-axis, but it excludes $x = 0$.

(ii) When $x = 0$, then $y = 0$ is the x-axis.

(iii) When $x < 0$, then $y = -1$ a straight line parallel to x-axis at a distance of 1 unit below x-axis, and it excludes $x = 0$.

The graph is exhibited in Fig. 2.23.

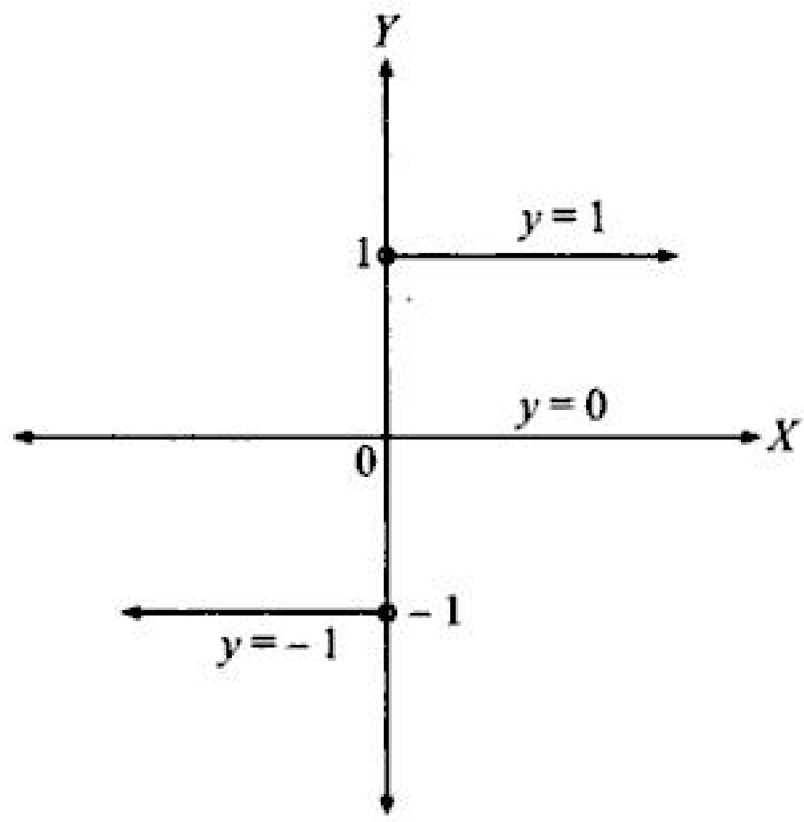

Fig. 2.23

26. Let $f(x) = ax + b$

$f(0) = 2 = a \times 0 + b, \quad \therefore b = 2.$

Also, $f(1) = -1 = a + 2, \quad \therefore a = -3$

27. As $f(xy) = f(x)\, f(y), \quad \therefore \quad f(0) = f(0.1) = f(0).\, f(1)$

$\Rightarrow \quad f(0)\,[f(1) - 1] = 0 \Rightarrow f(1) = 1$ as $f(0) \neq 0$

29. $$\frac{f(x) \times f(x^2)}{1 + [f(x)]^2} = \frac{\frac{1+x}{1-x} \times \frac{1+x^2}{1-x^2}}{1 + \left(\frac{1+x}{1-x}\right)^2} = \frac{1+x^2}{2(1+x^2)} = \frac{1}{2}$$

Also, $$f(f(x)) = \frac{1 + f(x)}{1 - f(x)} = \frac{1 + \frac{1+x}{1-x}}{1 - \frac{1+x}{1-x}} = -\frac{1}{x}$$

30. $f(a) + f(b) = \dfrac{b(a-a)}{b-a} + \dfrac{a(a-b)}{a-b} + \dfrac{b(b-a)}{b-a} + \dfrac{a(b-b)}{a-b} = a + b$

and $f(a + b) = \dfrac{b(a+b-a)}{b-a} + \dfrac{a(a+b-b)}{a-b} = \dfrac{a^2-b^2}{a-b} = a + b$

31. If $\alpha > 0$, then $|f(\alpha) - f(-\alpha)| = \left|\dfrac{|\alpha|}{\alpha} - \dfrac{|-\alpha|}{-\alpha}\right| = |1 - (-1)| = 2$

If $\alpha < 0$, then $|f(\alpha) - f(-\alpha)| = \left|\dfrac{|\alpha|}{\alpha} - \dfrac{|-\alpha|}{-\alpha}\right| = \left|\dfrac{-\alpha}{\alpha} - \dfrac{-\alpha}{-\alpha}\right| = 2$

32. $f(y) = \dfrac{1-y}{1+y} = \dfrac{1-(1-x)/(1+x)}{1+(1-x)/(1+x)} = \dfrac{2x}{1+x}\Big/\dfrac{2}{1+x} = x$

33. (a) $f(y) = (a - y^n)^{1/n} = [a - (a - x^n)]^{1/n}$ $\quad (\because\ y^n = a - x^n)$

$= (x^n)^{1/n} = x$

(b) Put $y = f(x)$ in (a) : $x = f(f(x))$

CHAPTER 3

TRIGONOMETRIC FUNCTIONS

Learning Objectives

After going through this chapter, the reader should be able to understand and appreciate:

- Measurement of Angles in Trigonometry
- Trigonometric Functions (circular functions) and their properties.
- Trigonometric Functions of Sum and Difference of Two Angles
- Trigonometric Equations

INTRODUCTION

The literal meaning of the word *Trigonometry* is the *science of triangle measurement*. The word *Trigonometry* is derived from two greek words *trigon* and *metron* which means measuring the sides of a triangle. It had its beginning more than two thousand years ago as a tool for astronomers. The Babylonians, Egyptians, Greeks and the Hindus studied trigonometry only because it helped them in unraveling the mysteries of the universe. In modern times, it has gained wider meaning and scope. Presently it is defined as that branch of mathematics which deals with the measurement of angles, whether of triangle or any other figure.

At present trigonometry is used in surveying, astronomy, navigation, physics and engineering.

In earlier classes, we have studied the trigonometric ratios of acute angles as the ratio of the sides of a right angled triangle. We have also studied the trigonometric identities and application of trigonometric ratios in solving the problems related to heights and distances. In this chapters, we will generalise the concept of trigonometric ratios to trigonometric functions and study their properties.

ANGLE

Let a revolving line starting from *OX*, revolve about its end point *O* in a plane in the direction of arrow and occupy the position *OP*, then it is said to trace out an angle *XOP*. Here *OX* is called the *initial position* and *OP*, the *terminal position*. The fixed point *O* is called *vertex*.

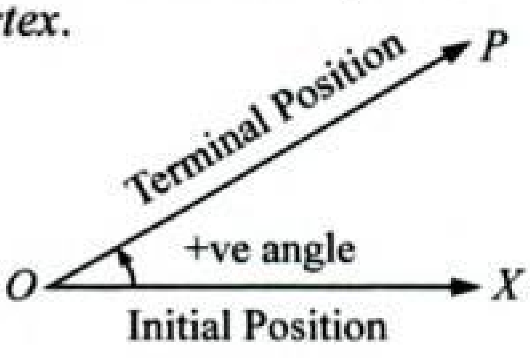

Fig. 3.1(a)

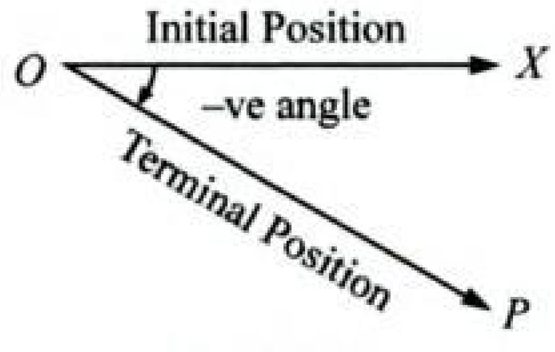

Fig. 3.1(b)

An angle is considered as the figure traced by rotating a given ray about its end point.

An angle XOP is said to be positive if it is traced out by a line revolving in the anti-clockwise direction as shown in Fig. 3.1(a) and negative if it is traced out by a line revolving in the clockwise direction as given in Fig. 3.1(b).

QUADRANTS

Let $X'OX$, $Y'OY$ be two perpendicular lines meeting in the point O. These divide the plane into four parts called *quadrants* (see Fig. 3.2).

(i) XOY is called the *first quadrant.*
(ii) YOX' is called the *second quadrant.*
(iii) $X'OY'$ is called the *third quadrant.*
(iv) $Y'OX$ is called the *fourth quadrant.*

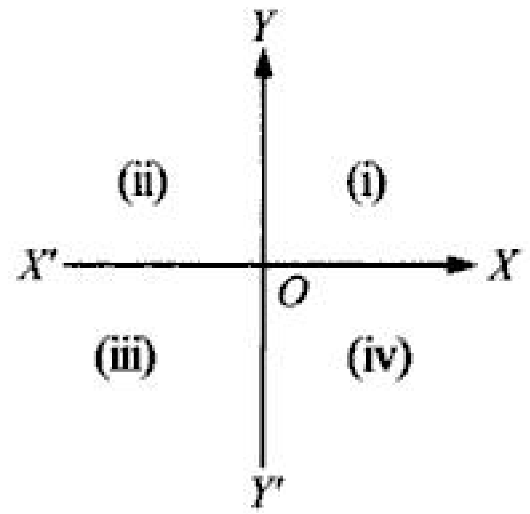

Fig. 3.2

ANGLE IN A PARTICULAR QUADRANT

If the final position of the revolving line tracing out an angle lies in a particular quadrant, the angle is said to lie in that quadrant.

MEASUREMENT OF AN ANGLE

We know that different units are used to measure the same quantity. For example, different units (kilogram, pound, etc.) are used to measure the same weight. In the same manner, different units are employed in the measurement of an angle in different systems. Here, we shall discuss two systems, viz. sexagesimal system and circular system.

Sexagesimal System

In this system an angle is measured in terms of degrees, minutes and seconds. We know that there are 4 right angles in a complete revolution and one complete revolution = 360 degrees (360°). Thus, we have

1 right angle = 90 degrees (90°)
1 degree = 60 minutes (60′)
1 minute = 60 seconds (60″)

Circular System

In this system, the angle is measured in radians.

Radian: Radian is the angle subtended at the centre of circle by an arc whose length is equal to the radius. Let O be the centre of a circle of radius r, cutoff an arc $AB = r$, then $\angle AOB = 1$ radian and is written as 1^c. (see Fig. 3.3).

Note: • 'c' used in the notation of radian is the first letter of the word *circular system.*

• When no unit is mentioned with an angle, it is always understood to be in radians.

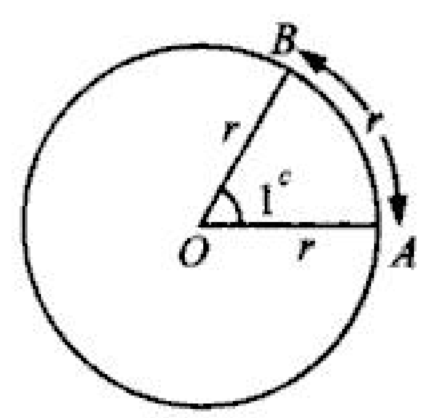

Fig. 3.3

To prove that radian is a constant angle:

Proof. Let O be the centre of a circle of radius r (Fig. 3.4). Let AB be the arc such that arc $AB = r$ = radius of the circle. Then $\angle AOB = 1$ radian. (By definition). Produce AO to meet the circumference in C. Since the angles at the centre of a circle are proportional to the arcs on which they stand, therefore,

$$\frac{\angle AOB}{\angle AOC} = \frac{\text{arc } AB}{\text{arc } AC} \Rightarrow \frac{1 \text{ radian}}{2 \text{ rt}. \angle s} = \frac{r}{\pi r} = \frac{1}{\pi}$$

[$\because \angle AOC = 2$ rt. $\angle s$ and arc $ABC = \frac{1}{2}$ circumference of the circle $= \frac{1}{2}(2\pi r) = \pi r$]

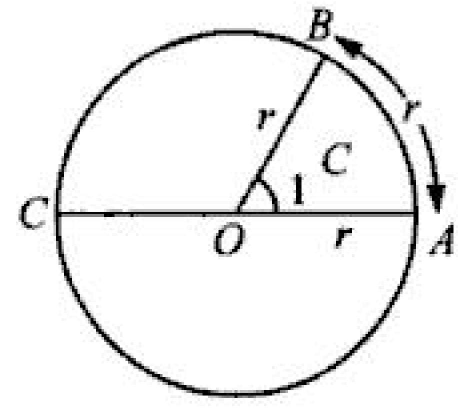

Fig. 3.4

$\therefore \quad 1 \text{ radian} = \frac{2\text{rt} \angle s}{\pi}$

which is independent of r, the radius of the circle. Hence radian is a constant angle.

Relation between Degree and Radian

Since a circle subtends at the centre an angle whose radian measure is 2π and its degree measure is 360°, it follows that

$$2\pi \text{ radian} = 360° \text{ or } \pi \text{ radian} = 180°$$

The above relation enables us to express a radian measure in terms of degree measure and a degree measure in terms of radian measure. Using approximate value of π as $\frac{22}{7}$, we have

$$1 \text{ radian} = \frac{180°}{\pi} = 57°16' \text{ (approximately)}$$

$$\text{Also } 1° = \frac{\pi}{180} \text{ radian} = 0.01746 \text{ radian (approximately)}$$

Arc-angle Relation

To prove that the number of radians in an angle subtended by an arc of a circle at the centre = arc/radius or $\theta = l/r$, where θ is the angle subtended by an arc.

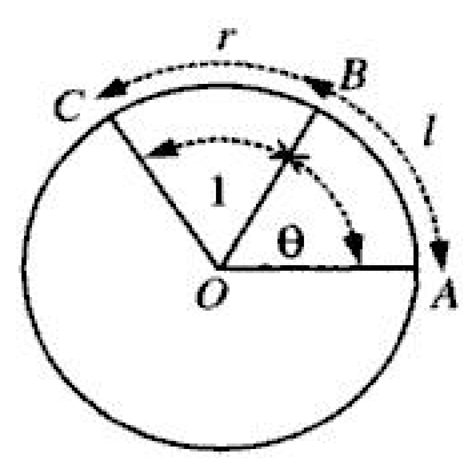

Fig. 3.5

Let O be the centre of the circle whose radius is r (Fig. 3.5). Let l be the length of the arc AB of the circle subtending an angle ($= \theta$ radians) at the centre O. Cutoff arc BC = radius. Join OC. Then $\angle BOC = 1$ radian, (by definition). Now, since angles at the centre of a circle are proportional to arcs subtending them,

$$\therefore \quad \frac{\angle AOB}{\angle BOC} = \frac{AB}{BC}, \text{ i.e., } \frac{\angle AOB}{1 \text{ radian}} = \frac{l}{r} \quad \text{or} \quad \angle AOB = \frac{l}{r} \text{ radians or } \theta = \frac{l}{r}$$

[$\because \angle AOB = \theta$ radians]

$$\therefore \quad \text{Number of radians in } \angle AOB = \frac{\text{Length of arc } AB}{\text{Radius of the circle}}$$

Circular Measure of an Angle

The circular measure of an angle is the number of radians it contains. An angle of one radian is denoted by 1^c.

Note:
- An angle can have any magnitude.
- In the result $\theta = l/r$; θ is always in radians and units of l and r are same.
- Following table gives the circular measure of some standard angles:

Angles in degree	0°	30°	45°	60°	90°	120°	135°	150°	180°	270°	360°
Circular measure	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

• The symbol π (read as pie) is the ratio of the circumference of the circle to its diameter. π is an irrational number and its value is generally taken as 22/7 unless otherwise mentioned.

Example 1. *In which quadrant do the following angles lie:* (i) 1190°; (ii) –1930°?

Solution: (i) Dividing 1190° by 360° (one complete revolution), we get

$$1190° = 3 \times 360° + 110°$$

Thus, the revolving line, after having made three complete revolutions in the *positive* (i.e., anti-clockwise) direction has further traced out an angle of 110° in the positive direction. Therefore, the angle is in the *second* quadrant.

(ii) Dividing –1930° by 360°, we get

$$-1930° = -5 \times 360° - 130°$$

Thus, the revolving line after having made five complete revolutions in the *negative* (i.e., clockwise) direction has further traced out an angle of 130° in the negative direction. Therefore, the angle is in the *third* quadrant.

Example 2. (i) *Find the radian measure corresponding to the following degree measure:* – 22° 30'

(ii) *Find the degree measure corresponding to the following radian measure:* 1/4

Solution: (i) Here the given angle = –22° 30'

$$= -\left(22\frac{1}{2}\right)^{\circ} = -\left(\frac{45}{2}\right)^{\circ} \quad \left[\because 30' = \left(\frac{30}{60}\right)^{\circ} = \left(\frac{1}{2}\right)^{\circ}\right]$$

$$\because \quad 180° = \pi \text{ radians} \Rightarrow 1° = \frac{\pi}{180} \text{ radians}$$

$$\therefore \quad -\left(22\frac{1}{2}\right)^{\circ} = \left[\frac{\pi}{180} \times \left(-\frac{45}{2}\right)\right] = -\frac{\pi}{8} \text{ radians}$$

Hence the circular measure of – 22° 30′ is $-\frac{\pi}{8}$.

(ii) We know that π radians = 180° $\therefore$ 1 radian = $\left(\frac{180}{\pi}\right)^{\circ}$

$$\frac{1}{4} \text{ radian} = \left(180 \times \frac{7}{22} \times \frac{1}{4}\right)^{\circ} = \left(\frac{315}{22}\right)^{\circ} = \left(14 + \frac{7}{22}\right)^{\circ} = 14° + \frac{7}{22} \times 60'$$

$$[\because 1° = 60']$$

$$= 14° + \left(\frac{210}{11}\right)' = 14° \ 19' \text{ approx.}$$

Hence the degree measure of $\frac{1}{4}$ radians is 14° 19′.

Example 3. *Express* 50° 37′ 30″ *in radians.*

Solution: $30'' = \frac{30'}{60} = \frac{1'}{2}$ $[\because 60'' = 1']$

$$37'30'' = 37\frac{1'}{2} = \frac{75'}{2} = \frac{75^\circ}{2\times 60} = \frac{5^\circ}{8} \qquad [\because 60' = 1^\circ]$$

$$\therefore \quad 50^\circ 37'30'' = 50\frac{5^\circ}{8} = \frac{405}{8}\times\frac{\pi}{180} \qquad [\because 180^\circ = \pi \text{ radians}]$$

$$= \frac{9\pi}{32} \text{ radians}$$

Example 4. *Find the degree measure corresponding to the following radian measures:*

(a) 1; (b) $\frac{\pi}{32}$.

Solution: Since π radians = 180°, i.e.,radian = $\frac{180^\circ}{\pi}$

(a) $$1 \text{ radian} = \frac{180^\circ}{\pi} = 180 \times \frac{7^\circ}{22} = \frac{630^\circ}{11}$$

$$= 57\frac{3^\circ}{11} = 57^\circ\frac{3}{11}\times 60' = 57^\circ\frac{180'}{11} = 57^\circ\, 16\frac{4'}{11}$$

$$= 57^\circ\, 16'\,\frac{4'}{11}\times 60'' = 57^\circ\, 16'\frac{240''}{11} = 57^\circ\, 16'22'' \text{ (approximately)}$$

(b) $$\frac{\pi}{32} \text{ radian} = \frac{\pi}{32}\times\frac{180^\circ}{\pi} = \frac{45^\circ}{8} = 5\frac{5^\circ}{8}$$

$$= 5^\circ\frac{5}{8}\times 60' = 5^\circ\frac{75'}{2} = 5^\circ\, 37\frac{1'}{2} = 5^\circ\, 37'\,\frac{1}{2}\times 60'' = 5^\circ\, 37'30''$$

Example 5. *The angles of a triangle are in A.P. The number of degrees in the least is to the number of radians in the greatest is as* $60{:}\pi$. *Find the angles in degrees.*

Solution: Let the three angles in A.P. be $a - d$, a, $a + d$. Since the sum of angles of a triangle = 180°

$$\therefore \quad a - d + a + a + d = 180^\circ \quad \text{or} \quad a = 60^\circ$$

$\therefore$ The angles are $(60 - d)^\circ$, 60°, $(60 + d)^\circ$. Again the number of degree in the least angle is $(60 - d)^\circ$ and greatest angle is $(60 + d)^\circ = (60 + d)\,\frac{\pi}{180}$ radians.

$\therefore$ By the given condition,

$$\frac{60 - d}{(60 + d)\,\frac{\pi}{180}} = \frac{60}{\pi} \quad \text{or} \quad \frac{60 - d}{60 + d} = \frac{1}{3}$$

or, $$180 - 3d = 60 + d \quad \text{or} \quad 4d = 120, \Rightarrow d = 30$$

Hence the angles are, 30°, 60°, 90°.

Example 6. *Express in circular measure and also in degrees the angle of*

(i) *a regular octagon;* (ii) *a regular polygon of* 40 *sides.*

Solution: (i) A regular octagon has 8 equal sides.

The sum of the 8 interior angles = 360° $\therefore$ Each interior $\angle = \frac{360^\circ}{8} = 45^\circ$

$\therefore$ Each interior angle = 180°, i.e., = 135°

$\because \quad 180° = \pi$ radians

$\therefore \quad 135° = \frac{\pi}{180} \times 135$ radians $= \frac{3\pi}{4}$ radians

Hence the angle of a regular octagon is $135° = \frac{3\pi}{4}$ radians.

(ii) Each exterior $\angle$ of a regular polygon of 40 sides $= \frac{360°}{40} = 9°$

$\therefore$ Each interior angle $= 180° - 9° = 171°$

Now, $180° = \pi$ radians

$\therefore \quad 171° = \frac{\pi}{180} \times 171$ radians $= \frac{19\pi}{20}$

Example 7. *Find the angle between the minute hand of a clock and the hour hand when the time is 7.20.*

Solution: When the time in a clock is 7.20, the minute hand is at mark 4 and the hour hand has crossed $\frac{20}{60} = \frac{1}{3}$rd of the angle between 7 and 8 (Fig. 3.6).

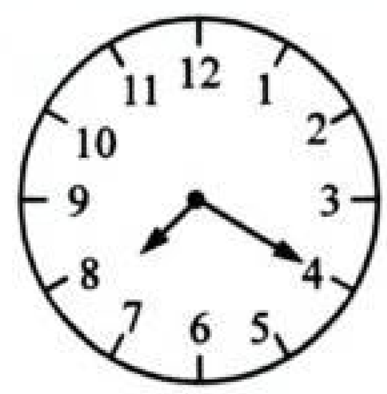

Fig. 3.6

Now, angle between two consecutive marks in a clock

$$= \left(\frac{360}{12}\right)^{\circ} = 30°$$

The required angle between the two hands at the time 7.20

$$= 3 \times 30° + \frac{1}{3}(30°) = 90° + 10° = 100°$$

Example 8. *Show that the minute hand of a watch gains 5° 30′ on the hour hand in a minute.*

Solution: Angle traced by the minute hand in 1 hour or 60 minutes $= 360°$

Angle traced by the minute hand in 1 minute $= \frac{360°}{60} = 6°$

Angle traced by the hour hand in 12 hours $= 360°$

Angle traced by the hour hand in 1 hour $= \frac{360°}{12} = 30°$

$\therefore$ Angle traced by the hour hand in 1 minute $= \frac{30°}{60} = \frac{1°}{2}$.

Difference in the angles traced by the minute and hour hands in 1 minute

$$= 6° - \frac{1}{2}^{\circ} = 5\frac{1}{2}^{\circ} = 5°\, 30'$$

Example 9. *Find the angle in degrees and radians of a regular pentagon.*

Solution: The sum of interior angles of a regular polygon of n sides $= (2n - 4)$ right angles.

$\therefore$ Interior angle of a regular polygon of n sides $= \left(\frac{2n-4}{n}\right)$ right angles

A pentagon has 5 sides.

$\therefore$ Interior angle of a regular pentagon $= \left(\dfrac{2\times 5-4}{5}\right)$ right angles

$$= \frac{6}{5} \text{ right angles} = \frac{6}{5} \times 90^\circ = 108^\circ$$

$\because$ $180^\circ = \pi$ radians

$\therefore$ $108^\circ = \dfrac{\pi}{180} \times 108$ radians $= \dfrac{3\pi}{5}$ radians

Example 10. *The difference between two acute angles of a right triangle is $\pi/9$. Find the angles in degrees.*

Solution: π radians = 180° $\quad\therefore\quad \dfrac{\pi}{9}$ radians $= \dfrac{\pi}{9} \times \dfrac{180^\circ}{\pi} = 20^\circ$

Let the two acute angles of the rt. Δ be x° and $y^\circ (x > y)$. Then $x + y = 90^\circ$ and $x - y = 20^\circ$

Solving these two equations,

$$2x = 110^\circ, \Rightarrow x = 110^\circ \div 2 = 55^\circ$$
$$2y = 70^\circ \Rightarrow y = 70^\circ \div 2 = 35^\circ.$$

Hence the two angles are 55° and 35°

Example 11. *The angles of a triangle are in the ratio 3 : 4 : 5, find the smallest angle in degree and the greatest angle in radians.*

Solution: Let the three angles be $3x$, $4x$ and $5x$ degrees, then

$$3x + 4x + 5x = 180^\circ \quad \therefore \quad x = 15^\circ$$

$\therefore$ Smallest angle = 45°

and Greatest angle $= 5x = 75^\circ = 75 \times \dfrac{\pi}{180} = \dfrac{5\pi}{12}$ radians

Example 12. *In a circle of 5 cm radius, what is the length of the arc which subtends an angle of 33° 15′ at the centre.*

Solution: Here $r = 5$ cm, $15' = \dfrac{15}{60} = \left(\dfrac{1}{4}\right)^\circ$

$\therefore$ $\theta = 33^\circ\, 15' = 33 + \dfrac{1}{4} = \dfrac{133}{4}$ degrees

$$= \frac{133}{4} \times \frac{\pi}{180} = \frac{133}{4} \times \frac{22}{7 \times 180} = \frac{1463}{2520} \text{ radians}$$

Now $\theta = \dfrac{l}{r} \;\therefore\; l = \theta r = \dfrac{1463}{2520} \times 5 = 2\dfrac{65}{72}$ cm. (approx.)

Example 13. *A cow is tied to a post by a rope. If the cow moves along a circular path keeping the rope always tight and describes 44 metres when it has traced out 72° at the centre, find the length of the rope.*

Solution: Here $l = 44$ m = length of the arc

$$\theta = 72^\circ = 72 \cdot \frac{\pi}{180} \text{ radians} = \frac{2\pi}{5} \text{ radians}$$

If r is the length of the rope, then

$$r = \frac{l}{\theta} = 44 \times \frac{5}{2\pi} = 110 \times \frac{7}{22} = 35 \text{ m} \qquad \left[\because \theta = \frac{l}{r}\right]$$

Example 14. *A railway train is moving on a circular curve of* 1500 *m radius at the rate of* 66 *km p.h. Through what angle has it turned in* 10 *seconds?*

Solution: Here $r = 1500$ m.

Distance moved in 1 hour (= 3600 seconds) = 66 km = 66000 m

$\therefore$ Distance moved in 10 seconds $= \dfrac{66000}{3600} \times 10$ m $= \dfrac{1100}{6}$ m

$\therefore$ l = length of the arc $= \dfrac{1100}{6}$ m

$$\therefore \quad \theta = \frac{l}{r} = \frac{1100}{6} \cdot \frac{1}{1500} = \frac{11}{90} \text{ radians} = \left(\frac{11}{90} \cdot \frac{180°}{\pi}\right) = \left(\frac{11}{90} \cdot 180 \cdot \frac{7}{22}\right)^{\circ} = 7°$$

Example 15. *The larger hand of a clock is* 28 *cm long. How many centimetres does its extremity move in* 20 *minutes.*

Solution: Here $r = 28$ cm. In 60 minutes, the larger hands moves 2π radians or 360°. In 20 minutes, it moves

$$\frac{2\pi}{60} \cdot 20 = \left(\frac{2\pi}{3}\right) \text{radians} \;\therefore\; \theta = \frac{2\pi}{3}$$

Now, $\quad l = r\theta = 28 \cdot \dfrac{22}{7} \cdot \dfrac{2}{3} = 58\dfrac{2}{3}$ cm $\quad [\because \theta = l/r]$

Example 16. *The wheel of a railway carriage is* 4 *ft in diameter and makes* 6 *revolutions in a second. How fast is the train going?*

Solution: Diameter of the wheel = 4 ft $\quad\therefore$ radius of the wheel = 2 ft

circumference of the wheel $= 2\pi r = 2\pi \times 2 = 4\pi$ ft

Number of the revolutions made in 1 second = 6

$\therefore$ Distance covered in 1 second $= 4\pi \times 6 = 24\pi$ ft

and Speed of the train $= 24\pi$ ft/sec

Example 17. Find the degree measure of the angle subtended at the centre of a circle (as shown in Fig. 3.7) of radius 100 cm by an arc of length 22 cm. $\left(\text{Use } \pi = \dfrac{22}{7}\right)$

Solution: We know that $l = r\theta$

where $\quad l$ = Length of arc = 20 cm

r = Radius of the circle

θ = Angle subtended at the centre

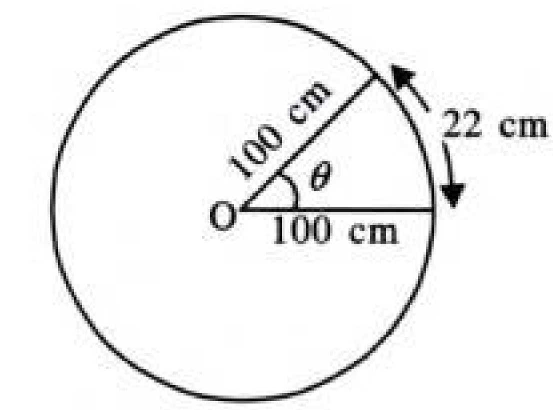

Fig. 3.7

$$\therefore \quad \theta = \frac{l}{r} = \frac{22}{100} = 0.22 \text{ radians}$$

$$= 0.22 \times \frac{180°}{\pi} \text{ degree}$$

$$= \left(\frac{0.22 \times 180 \times 7)}{22}\right)$$

$$= \frac{22}{100} \times \frac{180 \times 7}{22} = \frac{126}{10} = 12°36'$$

Example 18. *If, in two circles, arcs of the same length subtend angles of 60° and 75° at the centre, find the ratio of their radii.*

Solution:

Fig. 3.8

$$l = r_1 \theta = r_1 \times 60 \times \frac{\pi}{180} = \frac{\pi r_1}{3}$$

and

$$l = r_2 \theta = r_2 \times 75 \times \frac{\pi}{180} = \frac{5\pi r_2}{12}$$

Since l is same for both the circles (as shown in Fig. 3.8),

$$\Rightarrow \frac{\pi}{3} r_1 = \frac{5\pi}{12} r_2$$

$$\Rightarrow r_1 : r_2 = 5 : 4$$

Example 19. *Find the distance from the eye at which a coin of diameter 2 cm should be held so as just to conceal the full moon whose angular diameter is 31′.*

Solution: Consider Fig. 3.9. Let AB be the diameter of the moon and O, the eye of the observer so that $\angle AOB = 31'$. Let CD be the diameter of the coin. The full moon will be just concealed if the diameter of a coin also subtends the same angle as the diameter of the moon at O, i.e., if $\angle COD = 31'$. As $\angle COD$ is small, CD may be treated as the arc of a circle whose centre is O and radius = OC or OD, the distance of coin from O. Let $OC = OD = r$

Here l = Length of arc CD = 2 cm (nearly)

$$\theta = 31' = \frac{31}{60} \times \frac{\pi}{180} \text{ radians}$$

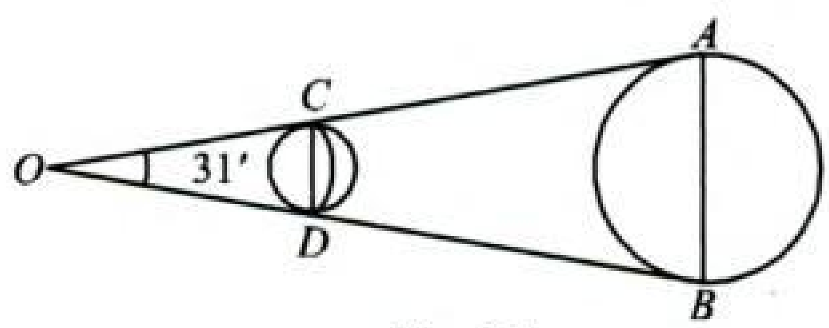

Fig. 3.9

Now, using $r = \frac{l}{\theta}$, we get $[\because l = r\theta]$

$$r = \frac{2 \times 60 \times 180}{31 \times \pi} = \frac{21600 \times 7}{31 \times 22} \qquad \left[\because \pi = \frac{22}{7}\right]$$

$$= \frac{75600}{341} \text{ cm} = \frac{756}{341} \text{ m} = 2.22 \text{ m (nearly)}$$

Example 20. *A wheel makes 360 revolutions in one minute. Through how many radians does it turn in one second?*

Solution: Angle rotated in one revolution = 2π radians

$\therefore$ Angle rotated in 360 revolutions = $360 \times 2\pi$ radians

$\Rightarrow$ Angle turned in one minute/60 sec = $360 \times 2\pi$

Hence, angle turned in 1 sec = $\dfrac{360 \times 2\pi}{60} = 12\pi$ radians

Example 21. *In a circle of diameter* 40 cm, *the length of a chord is* 20 cm. *Find the length of the minor arc of the chord.*

Solution: Refer Fig. 3.10. Since radius = length of chord = 20 cm

$\Rightarrow$ $\Delta\, OAB$ is equilateral triangle.

$\Rightarrow$ $\theta = 60°$

$\therefore$ $l = 20 \times 60° = 20 \times 60° \times \dfrac{\pi}{180°} = \dfrac{20\pi}{3}$

$l = \dfrac{20\pi}{3}$ cm

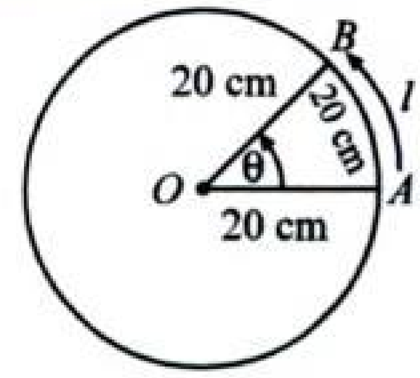

Fig. 3.10

Example 22. *Find the degree measure of the angle subtended at the centre of a circle of radius* 100 cm *by an arc of length* 22 cm *as shown in Fig.* 3.11. $\left(\text{Use } \pi = \dfrac{22}{7}\right)$.

Solution. We know that

$l = r\theta$

where l = Length of arc = 20 cm

r = Radius of circle = 100 cm

θ = Angle subtended at the centre

$\therefore$ $\theta = \dfrac{l}{r} = \dfrac{22}{100} = 0.22$ radians

$= 0.22 \times \dfrac{180°}{\pi}$ degree

$= \left(\dfrac{0.22 \times 180 \times 7}{22}\right)$

$= \left(\dfrac{22}{100} \times \dfrac{180 \times 7}{22}\right)$

$= \left(\dfrac{126}{10}\right) = 12°36'$

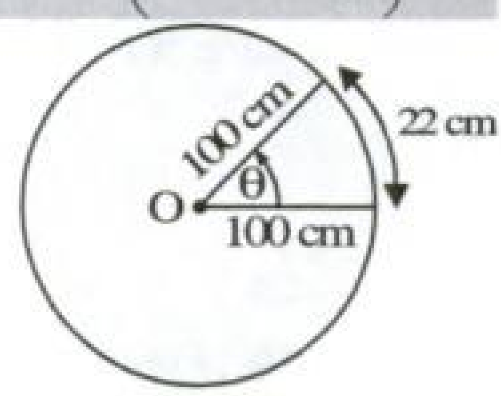

Fig. 3.11

12)126 (12 degree
120
6
×60
10)360 (36 minute
360
×

Example 23. *Find the angle in radian through which a pendulum swings if its length is* 75 cm *and the tip describes an arc of length:*

(i) 10 cm; (ii) 15 cm; (iii) 21 cm $\left(\text{Use } \pi = \dfrac{22}{7}\right)$

Solution: (i) $r = 75$

$l = 10$ cm

$\theta = ?$

$\therefore$ $\theta = \dfrac{l}{r} = \dfrac{10}{75}$ radians $= \dfrac{2}{15}$ radians

(ii) $r = 75$ cm

$l = 15$ cm

$\therefore$ $\theta = \dfrac{l}{r} = \dfrac{15}{75}$ radians $= \dfrac{1}{5}$ radians

(iii) $r = 75$

$l = 21$ cm

$\therefore \quad \theta = \frac{l}{r} = \frac{21}{75}$ radians $= \frac{7}{25}$ radians

EXERCISE 3.1

LEVEL OF DIFFICULTY A

1. Find the radian measure corresponding to the following degree measures:
   (i) 15°; (ii) 108°; (iii) 500°; (iv) 110°, 30′; (v) 45° 20′ 30″; (vi) – 36° 18′; (vii) 75°; (viii) 55° 15′; (ix) 25°; (x) –47°30′; (xi) 240°; (xii) 520°.
2. Find the degree measure corresponding to the following radian measures.
   (i) $\frac{7\pi}{3}$; (ii) $\frac{3\pi}{5}$; (iii) $\frac{17\pi}{9}$; (iv) $-\frac{7\pi}{10}$; (v) $\frac{1}{4}$; (vi) – 2; (vii) $\frac{1}{3}$; (viii) $\frac{2\pi}{9}$;
   (ix) $\frac{11}{16}$; (x) –4; (xi) $\frac{5\pi}{3}$; (xii) $\frac{7\pi}{6}$.
3. Find in radians the angle of a regular
   (i) pentagon; (ii) hexagon; (iii) decagon.
4. The difference between two acute angles of a right angled triangle is $\frac{3\pi}{10}$ radians. Find the angles in degrees.
5. Find the angle between the hour-hand and the minute-hand in circular measure at half past 4.
6. The angles of a triangle are in A.P., the greatest of them being 80°, find all the three angles in radians.
7. The angles of a quadrilateral are in A.P. and the greatest is double the least. Find the circular measure of the least.
8. The angles of a triangle are in A.P. The greatest angle is 5 times the least. Find the angles in circular measure.
9. The angles of a triangle are in A.P. and the number of radians in the greatest angle is to the number of degrees in the least one as π : 60. Find the angles in degrees.
10. In a circle of diameter 40 cm, the length of a chord is 20 cm. Find the length of the minor arc of the chord.
11. Find the angle (in degrees) through which a pendulum swings if its length is 50 cm and tip describes an arc of length
    (*a*) 10 cm; (*b*) 16 cm; (*c*) 26 cm $\left(\text{Use } \pi = \frac{22}{7}\right)$;
12. Find the length of the arc of a circle of radius 5 cm subtending a central angle measuring 15°.
13. A wheel 121 cm long is bent so as to lie along the arc of a circle of radius 180 cm. Find in degrees, the angle subtended at the centre by the wire.
14. A wire makes 180 revolutions in one minute. Through how many radians does it turn in one second?
15. Find the ratio of the radii of two circles at the centres of the which two equal arcs subtend angles of 30° and 63°.

16. A railway engine is moving along a circular railway track of radius 1500 metres with a speed of 66 km/hour. Find the angle turned by the engine in 10 seconds.

17. A circular wire of radius 3 cm is cut and bent so as to lie along the circumference of a hoop whose radius is 48 cm. Find in degrees the angle which is subtended at the centre of the hoop.

18. The angles of a polygon are in A.P. The smallest angle is $\frac{2\pi}{3}$ and the common difference is 5°. Find the number of sides of the polygon.

LEVEL OF DIFFICULTY B

19. A horse is tied to a post by a rope. If the horse moves along a circular path always keeping the rope tight and describes 88 metres when it has traced out 72° at the centre, find the length of the rope.

20. Meerut is 64 km from Delhi. Find the angle (in degrees) subtended at the centre of the earth by the arc joining these two towns, the earth being regarded as a sphere of 6336 km. radius.

21. Assuming that a person of normal eye-sight can read print at such a distance that the letters subtend an angle 5′ at his eye. Find what is the height of the letters that he can read at a distance of 2640 metres?

22. The angle in one regular polygon is to that in another is 3:2, and the number of sides in first is twice that in the second. Determine the number of sides of two polygons.

23. A railroad curve is to be laid out on a circle. What radius should be used if the track is to change direction by 25° in a distance of 40 metres.

24. The greatest angle of a cyclic quadrilateral is 3 times the least. Find all the angles if the other two are in the ratio 4:5.

25. The angles of a quadrilateral are in G.P. with common ratio $r > 1$. If the ratio of the largest angle to the smallest angle is 8°, find the angles in degrees.

26. The perimeter of a certain sector of a circle is equal to the length of the arc of a semi-circle having the same radius. Express the angle of the sector in degrees, minutes and seconds.

Answers

1. (i) $\frac{\pi}{12}$; (ii) $\frac{3\pi}{5}$; (iii) $\frac{25\pi}{9}$; (iv) $\frac{221\pi}{360}$;
(v) $\frac{5441\pi}{21600}$; (vi) $-\frac{121}{600}\pi$; (vii) $\frac{5\pi}{12}$; (viii) $\frac{221}{720}\pi$;
(ix) $\frac{5\pi}{36}$; (x) $-\frac{19\pi}{72}$; (xi) $\frac{4\pi}{3}$; (xii) $\frac{26\pi}{9}$;

2. (i) 420°; (ii) 108°; (iii) 340°; (iv) –126°.
(v) $14° \; 19' \; 5\frac{5''}{11}$; (vi) –144° 32′ 4″; (vii) 19° 5′ 27″; (viii) 40°.
(ix) 39° 22′ 30″; (x) 229° 5′ 27″ (nearly); (xi) 300°;
(xii) 210°.

3. (i) $\frac{3\pi}{5}$; (ii) $\frac{2\pi}{3}$; (iii) $\frac{4\pi}{5}$. **4.** 72°, 18°. **5.** $\frac{\pi}{4}$ radians.

6. $\frac{2\pi}{9}, \frac{\pi}{3}, \frac{4\pi}{9}$. **7.** $\frac{\pi}{3}$ radians **8.** $\frac{\pi}{9}, \frac{\pi}{3}, \frac{5\pi}{9}$.

9. 30°, 60° and 90°. **10.** $20\frac{20}{21}$ cm.

11. (a) 11° 27′; (b) 18° 20′; (c) 29° 47′. **12.** $\frac{5\pi}{12}$ cm. **13.** $38\frac{1^\circ}{2}$.

14. 6π. **15.** 21:10. **16.** 7°. **17.** 22.5°.

18. 9. **19.** 70 m. **20.** 34′43″ (nearly). **21.** 3.8 m (nearly).

22. 8, 4. **23.** 91.636 m. **24.** 45°, 80°, 100° 135°.

25. 24°, 48°, 96° and 192°. **26.** 65° 27′ 18″.

HINTS AND SOLUTIONS

2. (v) π radian = 180°

$$1 \text{ radian} = \frac{180^\circ}{\pi}$$

$$\frac{1}{4} \text{ radian} = \frac{180^\circ}{\pi} \times \frac{1}{4} \text{ degrees} = \frac{45 \times 7}{22} \text{ degrees} = \frac{315}{22} \text{ degrees} = 14^\circ, 19', 5\frac{5''}{11}$$

(vi) $$-2 \text{ radians} = -\frac{2 \times 180}{\pi} \text{ degrees} = -\frac{2 \times 180 \times 7}{22} \text{ degree} = -144^\circ\ 32'\ 4''.$$

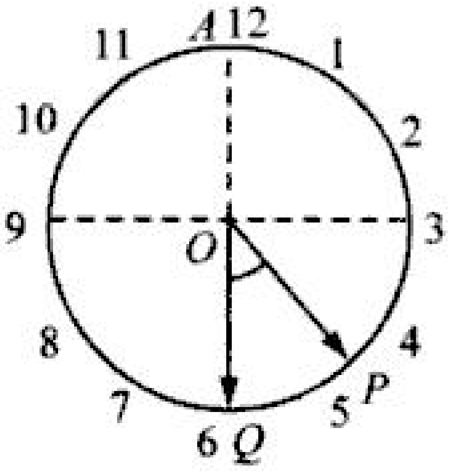

Fig. 3.12

5. Refer to Fig. 3.12. At half past 4, hour-hand will be at $4\frac{1}{2}$ and minute-hand will be at 6. In 12 hours, angle made by the hour hand is 60°.

$\therefore$ In 1 hour angle made by the hour-hand = 30°

$\therefore$ in $4\frac{1}{2}$ hours, i.e., $\frac{9}{2}$ hours angle made by the hour-hand = $\frac{9}{2} \times 30 = 135^\circ$

$\therefore$ $\angle AOP = 135^\circ$

In 60 minutes, angle made by the minute-hand = 360°

$\therefore$ In 1 minute, angle made by the minute-hand = 6°

$\therefore$ In 30 minutes angle made by the minute-hand = 6 × 30° = 180°

$\therefore$ $\angle AOQ = 180^\circ$

$$\therefore \angle POQ = \angle AOQ - \angle AOP = 180^\circ - 135^\circ = 45^\circ = 45 \times \frac{\pi}{180} = \frac{\pi}{4} \text{ radians.}$$

7. Let the four angles of the quadrilateral be $(a - 3d)^\circ$, $(a - d)^\circ$, $(a + d)^\circ$, $(a + 3d)^\circ$.

Their sum = $a - 3d + a - d + a + d + a + 3d = 360$ or $4a = 360$, $\therefore a = 90$

$\therefore$ The angles are $(90 - 3d)^\circ$, $(90 - d)^\circ$, $(90 + d)^\circ$, $(90 + 3d)^\circ$.

Greatest angle = $(90 + 3d)^\circ$

$\because$ Greatest $\angle$ = 2 × Least $\angle$ (Given)

$\therefore$ $90 + 3d = 2(90 - 3d)$ or $9d = 90$ $\therefore$ $d = 10$

$$\therefore \text{Least angle} = (90 - 3d)^\circ = (90 - 30)^\circ = 60^\circ = 60 \times \frac{\pi}{180} = \frac{\pi}{3} \text{ radians.}$$

9. Let the angles of the triangle be $(a - d)^\circ$, a° and $(a + d)^\circ$. Then,

$$(a - d) + a + (a + d) = 180^\circ \Rightarrow 3a = 180^\circ \Rightarrow a = 60^\circ$$

So, the angles are $(60 - d)^\circ$, 60°, $(60 + d)^\circ$. Clearly, $(60 - d)^\circ$ is the least angle and $(60 + d)^\circ$ is the greatest angle.

$$\text{Now, greatest angle} = (60 + d)^\circ = \left\{(60 + d)\frac{\pi}{180}\right\}^c$$

It is given that

$$\frac{\text{Number of degrees in the least angle}}{\text{Number of radians in the greatest angle}} = \frac{60}{\pi}$$

$$\Rightarrow \quad \frac{(60-d)}{(60+d)\dfrac{\pi}{180}} = \frac{60}{\pi} \Rightarrow 3(60-d) = (60+d) \Rightarrow 120 = 4d \Rightarrow d = 30$$

Hence, the angles are $(60-30)^\circ$, 60°, $(60+30)^\circ$, i.e., 30°, 60°, 90°.

10. Refer to Fig. 3.13. Radius $OA = OB = r = \frac{40}{2} = 20$ cm. Chord $AB = 20$ cm.

$\therefore$ ΔOAB is equilateral [$\because$ each side = 20 cm]

$$\theta = \angle AOB = 60^\circ = 60 \times \frac{\pi}{180} \text{ radians} = \frac{\pi}{3} \text{ radians.}$$

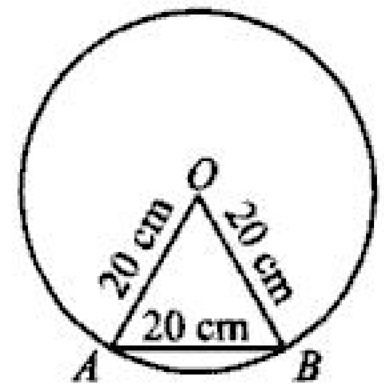

Fig. 3.13

$\therefore$ Length of the minor arc $AB = r\theta = 20 \times \frac{\pi}{3}$ cm $= \frac{20}{3} \times \frac{22}{7} = \frac{440}{21} = 20\frac{20}{21}$ cm.

14. No. of revolutions in one minute or 60 seconds = 180

$\therefore$ No. of revolutions in one second $= \frac{180}{60} = 3$

But angle turned in one revolution $= 360^\circ = 2\pi$ radians

$\therefore$ Angle turned by the wheel in one second $= 3 \times 2\pi = 6\pi$ radians

16. Radius of the circular path $(r) = 1500$ m

Distance covered in 1 hour (= 3600 seconds) = 66 km = 66000 m

Distance covered in 10 seconds $= \frac{66000}{3600} \times 10 = \frac{1100}{6}$ m

$\therefore$ l = Length of arc of circular path covered in 10 seconds $= \frac{1100}{6}$ m

Hence $\theta = \frac{l}{r} = \frac{1100}{6} \times \frac{1}{1500} = \frac{11}{90}$ radians

$$= \frac{11}{90} \times \frac{180^\circ}{\pi} = \frac{11}{90} \times 180 \times \frac{7}{22} = 22 \times \frac{7}{22} = 7^\circ$$

17. Length of arc = Circumference of circle of radius = 3 cm

$\therefore$ $l = 2\pi r = 2\pi \times 3 \text{ cm} = 6\pi$ cm

The radius of the hoop $(r) = 48$ cm

$$\theta = \frac{l}{r} = \frac{6\pi}{48} = \frac{\pi}{8} \text{ radians}$$

$\therefore$ Angle in degrees which is subtended at the centre of the hoop

$$= \frac{\pi}{8} \times \frac{180^\circ}{\pi} = \frac{45}{2} = 22.5^\circ$$

18. Let n be the number of sides of the polygon. Then the sum of the measures of the interior angles $= 2(n-2) \times 90^\circ$. But sum of n interior angles in $A.P.$

$$= \frac{n}{2}[2 \times 120^\circ + (n-1) \times 5^\circ] \qquad \left[\because \text{Sum of } n \text{ terms} = \frac{n}{2}[2a + (n-1)d]\right]$$

$\therefore \quad (2n-4) \times 90° = \frac{n}{2}[2 \times 120° + (n-1) \times 5°]$

$\Rightarrow \quad n^2 - 25n + 144 = 0 \Rightarrow (n-16)(n-9) = 0$

$\therefore \quad n = 16 \text{ or } 9$

But 16th interior angle = $120° + (16-1) \times 5° = 120° + 75° = 195° > 180°$, which is not possible.

Hence, the number of sides = 9.

19. Let O be the post, and A, B be the two positions of the horse. [see Fig. 3.14]. Here l, the length of the arc AB = 88 metres. Then

Angle subtended = $72° = 72 \times \frac{\pi}{180}$ radians = $\frac{2\pi}{5}$ radians

$\therefore \quad \theta = \frac{2\pi}{5}$ radians

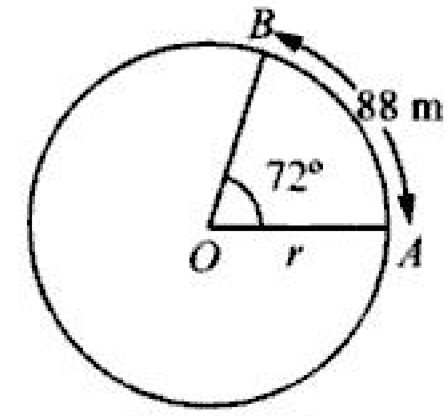

Fig. 3.14

If r (= length of the rope) be the radius of the circle. Then

$\theta = \frac{l}{r}$ gives $\frac{2\pi}{5} = \frac{88}{r} \Rightarrow r = 88 \times \frac{5}{2\pi}$ m $= 88 \times \frac{5}{2} \times \frac{7}{22} = 70$ metres

Hence the length of the rope = 70 metres.

20. Here $r = 6336$ km

$l = 64$ km

$\theta = \frac{l}{r} = \frac{64}{6336}$ radians $= \frac{1}{99} \times \frac{180°}{\pi}$

$= \frac{20}{11} \times \frac{7}{22}$ degrees

$= \frac{70}{121} \times 60 = 34'\frac{86}{121} \times 60'' = 34'\,43''$ (nearly).

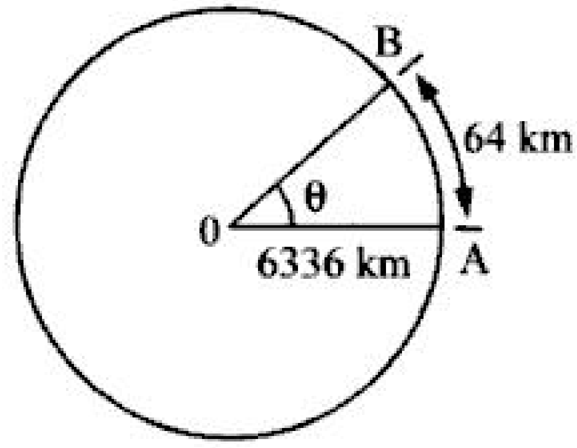

Fig. 3.15

21. Refer to Fig. 3.16. Let $AB = l$ be the height of the letter, and E, the eye of the observer so that $\angle AEB = 5'$. Since $\angle AEB$ is small, AB may be regarded as the arc of a circle with centre E and radius

$EA = EB = 2640$ metres $\quad \therefore \quad r = 2640$ metres

$\angle AEB = 5' = \frac{5}{60} \times \frac{\pi}{180}$ radians

$\therefore \quad \theta = \frac{5\pi}{60 \times 180}$

Now $\theta = \frac{l}{r} \Rightarrow \frac{5}{60} \times \frac{\pi}{180} = \frac{l}{2660 \text{ metres}}$

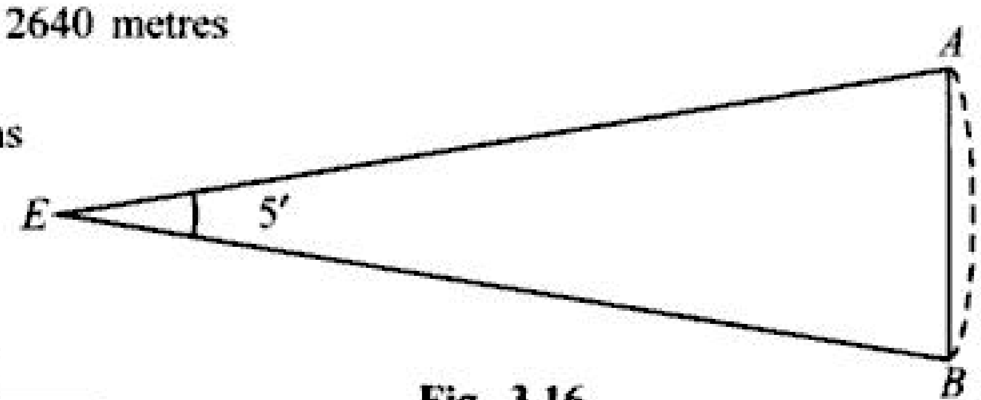

Fig. 3.16

$\Rightarrow l = \frac{5}{60} \times \frac{\pi}{180} \times 2640$ metres $= 3.8$ m (nearly).

23. Let the radius of the circle on which the railroad curve is to be laid down be x metres and the angle subtended by it at the centre is

$25°$ or $25 \times \left(\frac{\pi}{180}\right)$ radians $\quad \therefore \quad x\left(\frac{25\pi}{180}\right) = 40$

$\Rightarrow x = \frac{180 \times 40}{25\pi} = \frac{180 \times 40}{25} \times \frac{7}{22} = \frac{720 \times 7}{55} = \frac{144 \times 7}{11} = \frac{1008}{11} = 91.636$ metres

25. Let the angles be a, ar, ar^2, ar^3, where $a(1 + r + r^2 + r^3) = 360°$. Since $(ar^3/a) = 8°$, we get $r = 2°$. We obtain $a = 24°$. Hence, the angles are 24°, 48°, 96° and 192°.

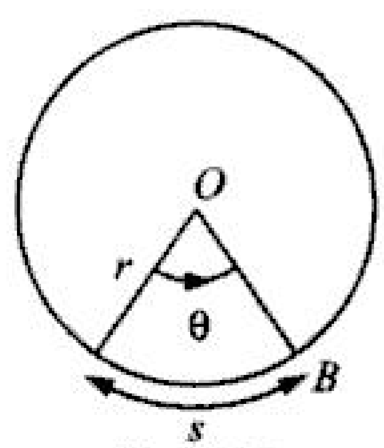

Fig. 3.17

26. We have perimeter of the sector $= r + r + s$

But $\theta = s/$
$r \Rightarrow s = \theta r$

$\therefore$ Perimeter of the sector $= r + r + r\theta \Rightarrow r(2 + \theta) = \pi r$ (given)

$$\Rightarrow 2 + \theta = \pi$$
$$\Rightarrow \theta = (\pi - 2) \text{ radians}$$
$$= (22/7 - 2) \text{ radians}$$
$$= \left(\frac{8}{7} \times \frac{180}{\pi}\right)^\circ = \left(\frac{8}{7} \times \frac{180}{22} \times 7\right)^\circ$$
$$= (65.455)^\circ$$
$$= 65° + 0.455°$$
$$= 65° + (0.455 \times 60)'$$
$$= 65° + (27.3)'$$
$$= 65° + 27' + (0.3)'$$
$$= 65° + 27' + (0.3 \times 60)''$$
$$= 65°27'18''.$$

CIRCULAR FUNCTIONS OR TRIGONOMETRIC FUNCTIONS OR TRIGONOMETRIC RATIOS (*t*-RATIOS)

Let a revolving line starting from OX trace out an angle $XOP = \theta$ in any of the four quadrants. Let M be the foot of perpendicular from P upon $X'OX$. Regarding OM and MP as directed lengths (OP always +ve), the ratios of OM, MP and OP with one another are called *circular functions* or trigonometrical ratios (briefly *t*-ratios) of the angle θ.

Let $OM = x$, $MP = y$ and $OP = r (r > 0)$, we define the various circular functions as follows:

(i) $\frac{MP}{OP} = \frac{y}{r}$ is called **sine of** θ, written as $\sin\theta$.

(ii) $\frac{OM}{OP} = \frac{x}{r}$ is called **cosine of** θ, written as $\cos\theta$.

(iii) $\frac{MP}{OM} = \frac{y}{x}$ $(x \neq 0)$ is called **tangent of** θ, written as $\tan\theta$.

(iv) $\frac{OM}{MP} = \frac{x}{y}$ $(y \neq 0)$ is called **cotangent of** θ, written as $\cot\theta$.

(v) $\frac{OP}{OM} = \frac{r}{x}$ $(x \neq 0)$ is called **secant of** θ, written as $\sec\theta$.

(vi) $\frac{OP}{MP} = \frac{r}{y}$ $(y \neq 0)$ is called **cosecant of** θ, written as $\text{cosec}\,\theta$.

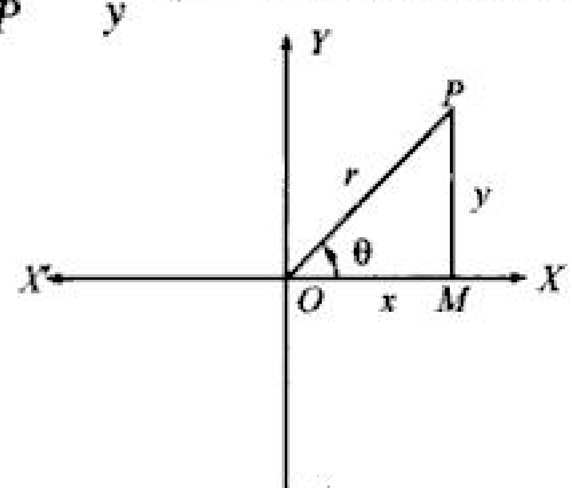

Fig. 3.18(a)

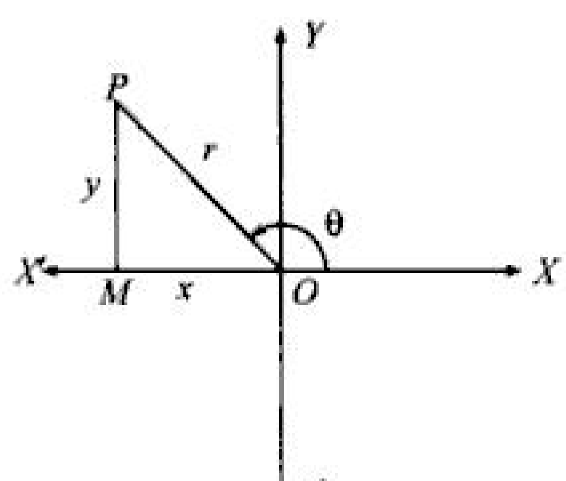

Fig. 3.18(b)

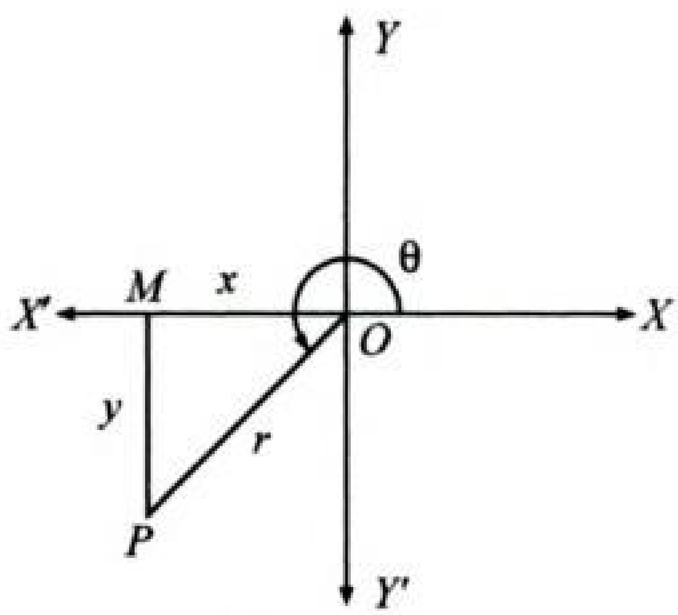

Fig. 3.18(c)

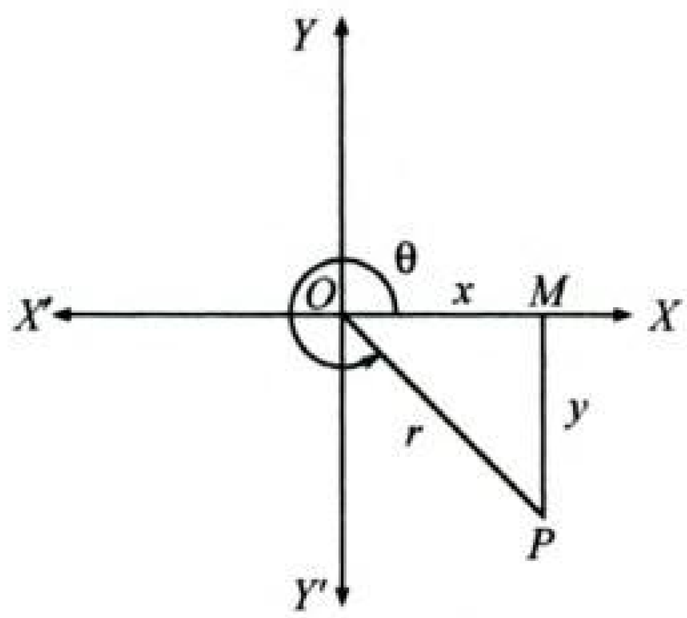

Fig. 3.18(d)

CAUTION

- sin θ does not mean sin $\times$ θ. It is a number denoting a certain ratio and not a product. sin θ stands for sine of the angle θ and briefly we rewrite it as sin θ. Similarly for other *t*-ratios.
- It should be noted that each of the *t*-ratios depends only upon the magnitude of the angle *XOP* and not upon the length of the revolving line *OP*. The *t*-ratios are always the same for the same angle, for if any point *P'* is taken on *OP* and *P'M'* is drawn $\perp$ *X'OX*, the right angled triangles *OMP* and *OM'P'* are similar and hence the ratios are same (Fig. 3.19).

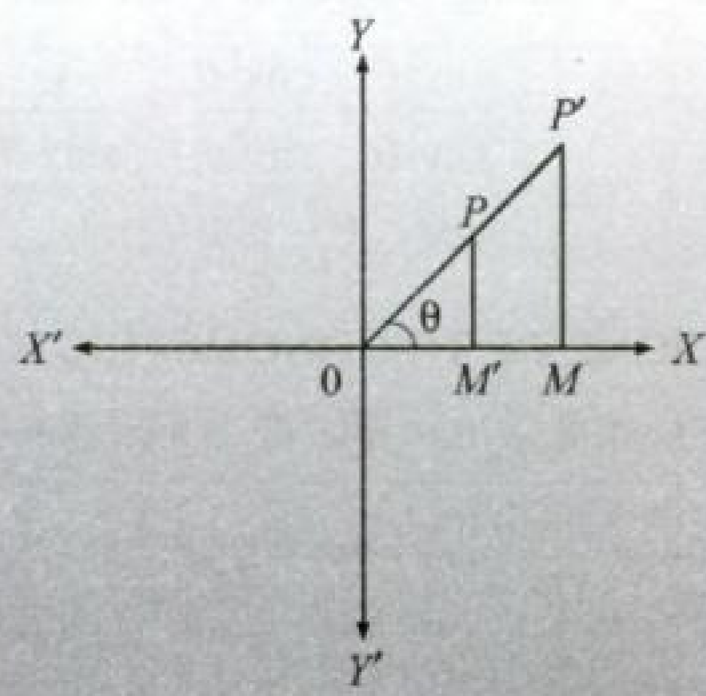

Fig. 3.19

Notation

$(\sin\theta)^2$ and $(\cos\theta)^2$ are written as $\sin^2\theta$ and $\cos^2\theta$. They are read as sin squared θ and cos squared θ respectively. Similarly, $(\sin\theta)^3$ is written as $\sin^3\theta$, read as sin cubed θ and so on. $(\sin\theta)^n$, read as sin θ to the power *n*, provided $(n \neq -1)$.

Cor.1. Relations between the trigonometric functions

From the definition of *t*-ratios, we have.

1. $\operatorname{cosec}\theta \times \sin\theta = \dfrac{r}{y}\times\dfrac{y}{r} = 1 \quad \therefore \quad \operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$ and $\sin\theta = \dfrac{1}{\operatorname{cosec}\theta}$

2. $\sec\theta \times \cos\theta = \dfrac{r}{x}\times\dfrac{x}{r} = 1 \quad \therefore \quad \sec\theta = \dfrac{1}{\cos\theta}$ and $\cos\theta = \dfrac{1}{\sec\theta}$

3. $\cot\theta \times \tan\theta = \dfrac{x}{y}\times\dfrac{y}{x} = 1 \quad \therefore \quad \cot\theta = \dfrac{1}{\tan\theta}$ and $\cos\theta = \dfrac{1}{\cot\theta}$

4. $\dfrac{\sin\theta}{\cos\theta} = \dfrac{y/r}{x/r} = \dfrac{y}{x} = \tan\theta$ and $\dfrac{\cos\theta}{\sin\theta} = \dfrac{x/r}{y/r} = \dfrac{x}{y} = \cot\theta$

   $\therefore\ \tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

Cor. 2. Three Fundamental Relations of *t*-ratios

1. $\sin^2\theta + \cos^2\theta = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \frac{y^2 + x^2}{r^2} = \frac{r^2}{r^2} = 1$

[$\because x^2 + y^2 = OM^2 + MP^2 = OP^2 = r^2$, By pythagoras theorem]

$\therefore \quad \sin^2\theta + \cos^2\theta = 1$ ...(1)

2. Dividing (1) by $\cos^2\theta$, we have

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} \Rightarrow \tan^2\theta + 1 = \sec^2\theta$$

$\therefore \quad \sec^2\theta - \tan^2\theta = 1$

3. Dividing (1) by $\sin^2\theta$, we have

$$\frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta} \Rightarrow 1 + \cot^2\theta = \operatorname{cosec}^2\theta$$

$\therefore \quad \operatorname{cosec}^2\theta - \cot^2\theta = 1$

SIGNS OF TRIGONOMETRIC RATIOS IN DIFFERENT QUADRANTS

The following table describes the signs of various *t*-ratios in different quadrants.

Quadrant	I	II	III	IV
$MP = y$	$+ve$	$+ve$	$-ve$	$-ve$
$OM = x$	$+ve$	$-ve$	$-ve$	$+ve$
$\sin\theta = \frac{y}{r}$	$\frac{+ve}{+ve} = +ve$	$\frac{+ve}{+ve} = +ve$	$\frac{-ve}{+ve} = -ve$	$\frac{-ve}{+ve} = -ve$
$\cos\theta = \frac{x}{r}$	$\frac{+ve}{+ve} = +ve$	$\frac{-ve}{+ve} = -ve$	$\frac{-ve}{+ve} = -ve$	$\frac{+ve}{+ve} = +ve$
$\tan\theta = \frac{y}{x}, x \neq 0$	$\frac{+ve}{+ve} = +ve$	$\frac{+ve}{-ve} = -ve$	$\frac{-ve}{-ve} = +ve$	$\frac{-ve}{+ve} = -ve$

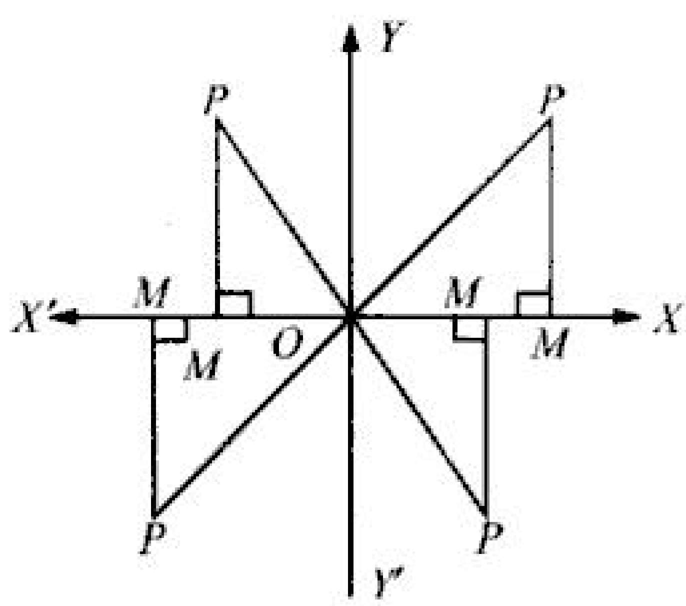

Fig. 3.20

The signs of other *t*-ratios can be found by using reciprocal relations, i.e.,

$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$, $\sec\theta = \frac{1}{\cos\theta}$ and $\cot\theta = \frac{1}{\tan\theta}$. So, we have

Quadrant: →	I	II	III	IV
t-ratios which are +*ve*	All	sin θ cosec θ	tan θ cot θ	cos θ sec θ

IMPORTANT

(i) In the first quadrant, **all** are **+ve**.
(ii) In the second quadrant, **sin** and **cosec** are **+ve**
(iii) In the third quadrant, **tangent** and **cotangent** are **+ve**
(iv) In the fourth quadrant, **cosine** and **secant** are **+ve**.

Simple Rule to Remember:

add – sugar – to – coffee
or
after – school – to – college

In above, 'a' stands for 'all', 's' stands for 'sine', 't' stands for 'tan' and 'c' stands for 'cos'. The reciprocal of these ratios are also positive in the respective quadrants.

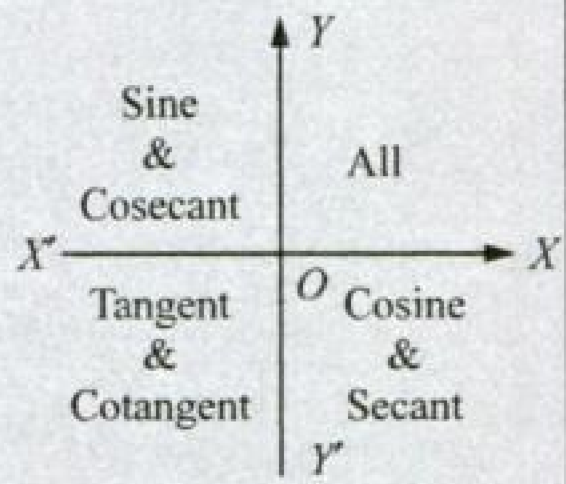

Fig. 3.21

TRIGONOMETRIC RATIOS OF STANDARD ANGLES

In this section, we shall find *t*-ratios of five standard angles 0°, 30°, 45°, 60° and 90°.

T-ratios of 45° or π/4

Make an angle $XOP = 45°$. Draw $PM \perp OX$. Then

$\angle OPM = \angle MOP = 45°$

∴ ΔOPM is isosceles.

∴ $MP = OM = a$(say)

∴ $OP = \sqrt{a^2 + a^2} = a\sqrt{2}$

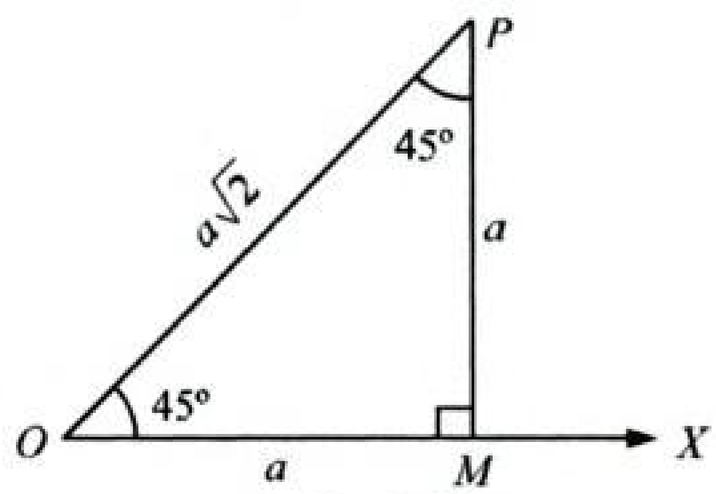

Fig. 3.22

Thus, in the right angled $\Delta\, OMP$,

$$\sin 45° = \frac{MP}{OP} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\cos 45° = \frac{OM}{OP} = \frac{a}{a\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\tan 45° = \frac{MP}{OM} = \frac{a}{a} = 1$$

$$\cot 45° = \frac{OM}{MP} = \frac{a}{a} = 1$$

$$\sec 45° = \frac{OP}{OM} = \frac{a\sqrt{2}}{a} = \sqrt{2}$$

$$\operatorname{cosec} 45° = \frac{OP}{MP} = \frac{a\sqrt{2}}{a} = \sqrt{2}$$

T-ratios of 30° or $\pi/6$:

Make an angle $XOP = 30°$. Draw $PM \perp OX$, as shown in Fig. 3.23. Therefore, in the right-angled ΔOMP, side opposite to 30° is half of the hypotenuse, i.e.,

$$MP = \frac{1}{2} OP$$

Let $MP = a$, then $OP = 2a$

$$\therefore \quad OM = \sqrt{4a^2 - a^2} = a\sqrt{3}$$

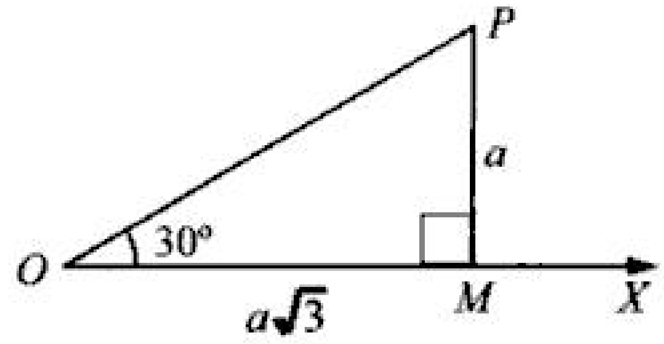

Fig. 3.23

Thus, in the right angled ΔOMP,

$$\sin 30° = \frac{MP}{OP} = \frac{a}{2a} = \frac{1}{2}$$

$$\cos 30° = \frac{OM}{OP} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\tan 30° = \frac{MP}{OM} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\cot 30° = \frac{OM}{MP} = \frac{a\sqrt{3}}{a} = \sqrt{3}$$

$$\sec 30° = \frac{OP}{OM} = \frac{2a}{a\sqrt{3}} = \frac{2}{\sqrt{3}}$$

$$\text{cosec } 30° = \frac{OP}{MP} = \frac{2a}{a} = 2$$

T-ratios of 60° or $\pi/3$:

Refer to Fig. 3.24. Make an angle $P'OP = 60°$. Draw $PM \perp OP'$. Then $\angle OPM = 30°$. Thus, side opposite to 30° is half the hypotenuse i.e., $OM = (1/2)\, OP$.

Let OM, $= a$, then $OP = 2a$

$$\therefore \quad MP = \sqrt{4a^2 - a^2} = a\sqrt{3}$$

Thus, in the right angled ΔOPM,

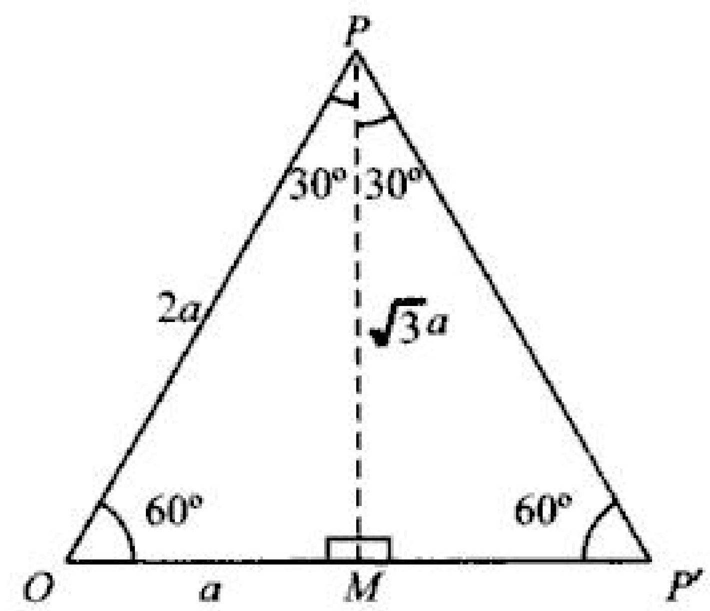

Fig. 3.24

$$\sin 60° = \frac{MP}{OP} = \frac{a\sqrt{3}}{2a} = \frac{\sqrt{3}}{2}$$

$$\cos 60° = \frac{OM}{OP} = \frac{a}{2a} = \frac{1}{2}$$

$$\tan 60° = \frac{MP}{OM} = \frac{a\sqrt{3}}{a} = \sqrt{3}$$

$$\cot 60° = \frac{OM}{MP} = \frac{a}{a\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\sec 60° = \frac{OP}{OM} = \frac{2a}{a} = 2$$

$$\text{cosec } 60° = \frac{OP}{MP} = \frac{2a}{a\sqrt{3}} = \frac{2}{\sqrt{3}}$$

T-ratios of 0°:

Make $\angle XOP = \theta$ (Fig. 3.25) as small as possible. Draw $PM \perp OX$. Now, θ becomes zero, when M and P coincide.

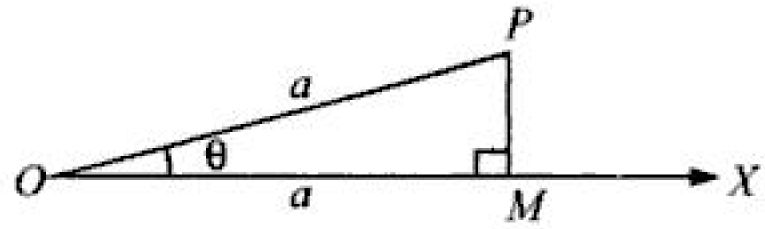

Fig. 3.25

i.e., $MP = 0$ and $OM = OP = a$ (say)

Thus, in the right angled $\Delta\, OMP$,

$$\sin 0° = \frac{MP}{OP} = \frac{0}{a} = 0$$

$$\cos 0° = \frac{OM}{OP} = \frac{a}{a} = 1$$

$$\tan 0° = \frac{MP}{OM} = \frac{0}{a} = 0$$

$$\cot 0° = \frac{OM}{MP} = \frac{a}{0} = \infty$$

$$\sec 0° = \frac{OP}{OM} = \frac{a}{a} = 1$$

$$\text{cosec } 0° = \frac{OP}{MP} = \frac{a}{0} = \infty$$

T-ratios of 90° or $\pi/2$:

Refer to Fig. 3.26. Make $\angle XOP = \theta$, nearly equal to 90°. Draw $PM \perp OX$. Now, θ becomes 90° when O and M coincide.

i.e., $OM = 0$ and $OP = MP = a$ (say)

Thus, in the right angled $\Delta\, OMP$,

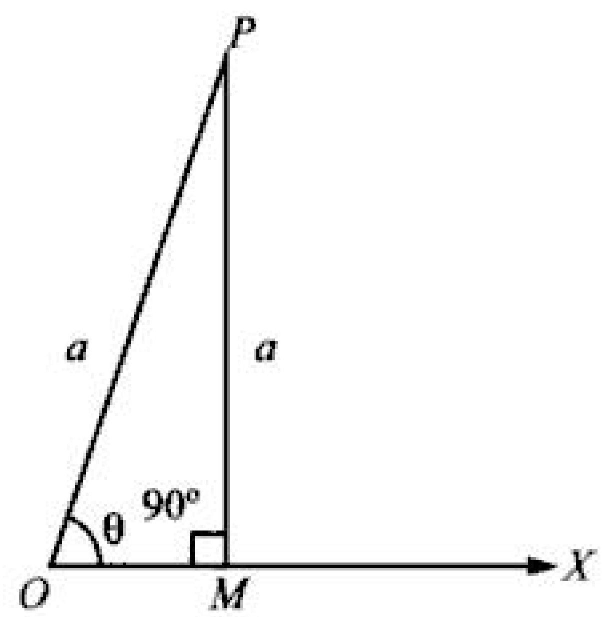

Fig. 3.26

$$\sin 90° = \frac{MP}{OP} = \frac{a}{a} = 1$$

$$\cos 90° = \frac{OM}{OP} = \frac{0}{a} = 0$$

$$\tan 90° = \frac{MP}{OM} = \frac{a}{0} = \infty$$

$$\cot 90° = \frac{OM}{MP} = \frac{0}{a} = 0$$

$$\sec 90° = \frac{OP}{OM} = \frac{a}{0} = \infty$$

$$\text{cosec } 90° = \frac{OP}{MP} = \frac{a}{a} = 1.$$

Remark. All these values of *t*-ratios can be put in the tabular form which the students must remember.

t-ratio / angle(θ)	0°	30°	45°	60°	90°
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	Undefined

The values of $\cot\theta$, $\sec\theta$ and $\text{cosec}\,\theta$ can be found from the above table by using the relations $\cot\theta = \frac{\cos\theta}{\sin\theta}$; $\sec\theta = \frac{1}{\cos\theta}$ and $\text{cosec}\,\theta = \frac{1}{\sin\theta}$.

Values of $\cos\theta$ and $\sin\theta$ for $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π

With O as centre and radius 1 unit, draw a circle cutting the coordinate axes at A, B, A' and B', as shown in Fig. 3.27. Let $P(x, y)$ be any point on this circle. If $\angle AOP = \theta$, then $\sin\theta = y$ and $\cos\theta = x$. At A, $\theta = 0$, $x = 1$, $y = 0$ $\therefore$ $\cos\theta = 1$ and $\sin\theta = 0$.

At B, $\theta = \frac{\pi}{2}$, $x = 0$, $y = 1$

$\therefore$ $\cos\frac{\pi}{2} = 0$ and $\sin\frac{\pi}{2} = 1$

At A', $\theta = \pi$, $x = -1$, $y = 0$

$\therefore$ $\cos\pi = -1$ and $\sin\pi = 0$

At B', $\theta = \frac{3\pi}{2}$, $x = 0$, $y = -1$

$\therefore$ $\cos\frac{3\pi}{2} = 0$ and $\sin\frac{3\pi}{2} = -1$

At A, $\theta = 2\pi$, $x = 1$, $y = 0$

$\therefore$ $\cos 2\pi = 1$ and $\sin 2\pi = 0$

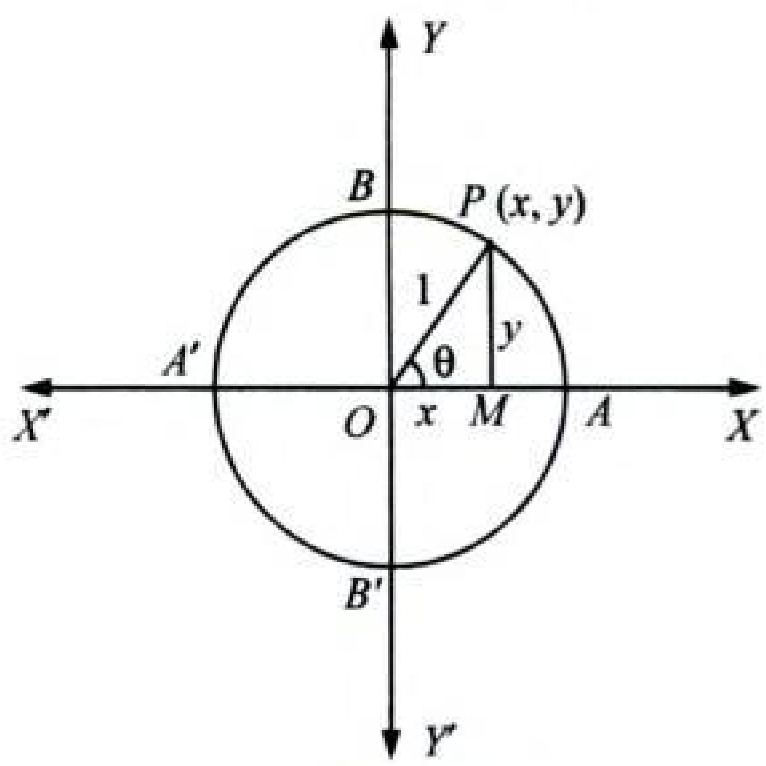

Fig. 3.27

Example 1. *If* $\tan\theta = \frac{24}{7}$ *and* $\pi < \theta < \frac{3\pi}{2}$, *find the values of* $\sin\theta$ *and* $\cos\theta$.

Solution: We know that

$$\sec^2\theta = 1 + \tan^2\theta = 1 + \left(\frac{24}{7}\right)^2 \qquad \left[\text{Putting } \tan\theta = \frac{24}{7}\right]$$

$$= 1 + \frac{576}{49}$$

$$\therefore \quad \sec^2\theta = \frac{625}{49} = \left(\frac{25}{7}\right)^2$$

$\therefore \quad \cos^2\theta = \left(\frac{7}{25}\right)^2$ ...(1)

$\therefore \quad \cos\theta = \pm\frac{7}{25}$

As $\cos\theta$ is $-ve$ in 3rd quadrant, $\therefore \cos\theta = -\frac{7}{25}$

Now, $\sin^2\theta = 1 - \cos^2\theta$

$$= 1 - \left(\frac{7}{25}\right)^2 = \frac{576}{625} = \left(\frac{24}{25}\right)^2$$

$\therefore \quad \sin\theta = \pm\frac{24}{25}$

As $\sin\theta$ is $-ve$ in 3rd quadrant, $\therefore \sin\theta = -\frac{24}{25}$

Example 2. *Find the values of the other five trigonometric functions in each of the following problems:*

(i) $\cos\theta = -\frac{1}{2}$, *$\theta$ in quadrant* II;

(ii) $\tan\theta = \frac{3}{4}$, *θ in quadrant* III;

(iii) $\sin\theta = \frac{3}{5}$, *θ in quadrant* I.

Solution: In the second quadrant, $\sin\theta$ and $\operatorname{cosec}\theta$ are positive and $\cos\theta$, $\sec\theta$, $\tan\theta$ and $\cot\theta$ are negative.

Now, $\cos\theta = -\frac{1}{2}$ (Given)

$\therefore \quad \sec\theta = \frac{1}{\cos\theta} = -2$

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \left(-\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$$

$\therefore \quad \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \operatorname{cosec} = \frac{1}{\sin\theta} = \frac{2}{\sqrt{3}}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3} \Rightarrow \cot\theta = \frac{1}{\tan\theta} = -\frac{1}{\sqrt{3}}.$$

(ii) θ lies in the third quadrant and in third quadrant $\tan\theta$ and $\cot\theta$ are positive and $\sin\theta$, $\cos\theta$, $\sec\theta$ and $\operatorname{cosec}\theta$ are negative.

$\tan\theta = \frac{3}{4}$ (Given)

$\Rightarrow \quad \cot\theta = \frac{1}{\tan\theta} = \frac{4}{3}$

$$\sec^2\theta = 1 + \tan^2\theta = 1 + \left(\frac{3}{4}\right)^2 = 1 + \frac{9}{16} = \frac{25}{16}$$

$$\Rightarrow \quad \sec\theta = -\frac{5}{4} \Rightarrow \cos\theta = \frac{1}{\sec\theta} = -\frac{4}{5}$$

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{16}{25} = \frac{9}{25}$$

$$\therefore \quad \sin\theta = -\frac{3}{5} \Rightarrow \text{cosec}\theta = \frac{1}{\sin\theta} = -\frac{5}{3}$$

(iii) Here θ lies in the first quadrant and in first quadrant, all the six *t*-ratios are positive. We have

$$\sin\theta = \frac{3}{5} \qquad \text{(Given)}$$

$$\Rightarrow \quad \text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{1}{3/5} = \frac{5}{3}$$

$$\cos^2\theta = 1 - \sin^2\theta = 1 - \frac{9}{25} = \frac{16}{25}$$

$$\therefore \quad \cos\theta = \frac{4}{5} \Rightarrow \sec\theta = \frac{1}{\cos\theta} = \frac{5}{4}$$

$$\tan\theta = \frac{\sin 6}{\cos\theta} = \frac{3/5}{4/5} = \frac{3}{4} \Rightarrow \cot\theta = \frac{1}{\tan\theta} = \frac{1}{3/4} = \frac{4}{3}$$

Example 3. *Find the value of*

$$3\tan^2 45° - \sin^2 60° - \frac{1}{2}\cot^2 30° + \frac{1}{8}\sec^2 45°$$

Solution: $3\tan^2 45° - \sin^2 60° - \frac{1}{2}\cot^2 30° + \frac{1}{8}\sec^2 45°$

$$= 3(1)^2 - \left(\frac{\sqrt{3}}{2}\right)^2 - \frac{1}{2}(\sqrt{3})^2 + \frac{1}{8}(\sqrt{2})^2 = 3 - \frac{3}{4} - \frac{3}{2} + \frac{2}{8}$$

$$= 3 - \frac{3}{4} - \frac{3}{2} + \frac{1}{4} = \frac{12 - 3 - 6 + 1}{4} = \frac{4}{4} = 1.$$

Example 4. *If* $\sin(A + B) = \frac{\sqrt{3}}{2}$, $\cos(A - B) = \frac{\sqrt{3}}{2}$, *then find A and B if they lie in the first quadrant.*

Solution: $\sin(A + B) = \frac{\sqrt{3}}{2} = \sin 60°$

$$\therefore \quad A + B = 60° \qquad ...(1)$$

and $\quad \cos(A - B) = \frac{\sqrt{3}}{2} = \cos 30°$

$$\therefore \quad A - B = 30° \qquad ...(2)$$

Solving (1) and (2), we get $A = 45°$ and $B = 15°$.

Example 5. *If* $\sec\theta = \sqrt{2}$ *and* $\frac{3\pi}{2} < \theta < 2\pi$, *find the value of* $\frac{1+\tan\theta+\operatorname{cosec}\theta}{1+\cot\theta-\operatorname{cosec}\theta}$

Solution: $\sec\theta = \sqrt{2}$

$\therefore$ $\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\sqrt{2}}$ $\therefore$ $\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{1}{2} = \frac{1}{2}$

Thus, $\sin\theta = \pm\frac{1}{\sqrt{2}}$

Now, $\frac{3\pi}{2} < \theta < 2\pi \Rightarrow \theta$ lies in the fourth quadrant $\therefore$ $\sin\theta$ is negative

$$\Rightarrow \sin\theta = -\frac{1}{\sqrt{2}} \quad \therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{-1/2}{1/\sqrt{2}} = -1$$

$$\therefore \quad \cot\theta = \frac{1}{\tan\theta} = -1 \text{ and } \operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{-1}{\sqrt{2}}} = -\sqrt{2}$$

$$\therefore \quad \frac{1+\tan\theta+\operatorname{cosec}\theta}{1+\cot\theta-\operatorname{cosec}\theta} = \frac{1+(-1)+(-\sqrt{2})}{1+(-1)-(-\sqrt{2})} = -\frac{\sqrt{2}}{\sqrt{2}} = -1.$$

Example 6. *If* θ *is the positive acute angle, solve the equation* $4\cos^2\theta - 4\sin\theta = 1$.

Solution: $4\cos^2\theta - 4\sin\theta = 1$ or $4(1 - \sin^2\theta) - 4\sin\theta = 1$

or $4\sin^2\theta + 4\sin\theta - 3 = 0$

which is a quadratic in $\sin\theta$.

$$\sin\theta = \frac{-4 \pm \sqrt{(4)^2 - 4.4(-3)}}{2.4} = \frac{-4\pm 8}{8} = \frac{1}{2} \text{ or } -\frac{3}{2}$$

As $\sin\theta \le 1$ numerically, hence the value of $\sin\theta = -\frac{3}{2}$ is not possible.

$\therefore$ $\sin\theta = \frac{1}{2} = \sin 30°$ $\therefore$ $\theta = 30°$.

Example 7. *If A, B, C are acute positive angles satisfying the equations* $\sin(B + C - A) = 1$, $\cos(C + A - B) = 1$ *and* $\tan(A + B - C) = 1$, *find the values of A, B and C.*

Solution: Given $\sin(B + C - A) = 1 = \sin 90°$ $\therefore$ $B + C - A = 90°$ ...(1)

$\cos(C + A - B) = 1 = \cos 0°$ $\therefore$ $C + A - B = 0°$ ...(2)

$\tan(A + B - C) = 1 = \tan 45°$ $\therefore$ $A + B - C = 45°$ ...(3)

Adding (1) and (2), we get $2C = 90° \Rightarrow C = 45°$

Adding (1) and (3), we get $2B = 135° \Rightarrow B = 67\frac{1°}{2}$

Adding (2) and (3), we get $2A = 45° \Rightarrow A = 22\frac{1°}{2}$

Hence $A = 22\frac{1°}{2}$, $B = 67\frac{1°}{2}$, $C = 45°$ are the required values.

Example 8. *If* $\sin A = 3/5$, $\tan B = 1/2$ *and* $\pi/2 < A < \pi < B < 3\pi/2$, *find the value of* $8\tan A - \sqrt{5}\sec B$.

Solution: $\cos^2 A = 1 - \sin^2 A = 1 - \frac{9}{25} = \frac{16}{25}$ $\quad\therefore\quad \cos A = \pm\frac{4}{5}$

$$\sec^2 B = 1 + \tan^2 B = 1 + \frac{1}{4} = \frac{5}{4} \quad\therefore\quad \sec B = \pm\frac{\sqrt{5}}{2}$$

Now, $\frac{\pi}{2} < A < \pi$, therefore, A lies in the second quadrant. $\cos A$ is negative, Hence $\cos A = -\frac{4}{5}$.

$$\therefore\ \tan A = \frac{\sin A}{-\cos A} = \frac{3/5}{-4/5} = -\frac{3}{4}$$

Also $\pi < B < \frac{3\pi}{2}$, therefore, B lies in the third quadrant. $\sec B$ is negative. Hence $\sec B = -\frac{\sqrt{5}}{2}$.

$$\therefore\quad 8\tan A - \sqrt{5}\sec B = 8\left(\frac{-3}{4}\right) - \sqrt{5}\left(-\frac{\sqrt{5}}{2}\right) = -6 + \frac{5}{2} = \frac{-7}{2}$$

Example 9. *If* θ *is a positive acute angle, solve the equation* $3\tan\theta + \cot\theta = 5\operatorname{cosec}\theta$.

Solution: $3\tan\theta + \cot\theta = 5\operatorname{cosec}\theta \Rightarrow 3\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta} = \frac{5}{\sin\theta}$

Multiplying by $\cos\theta\sin\theta$, we get

$$3\sin^2\theta + \cos^2\theta = 5\cos\theta \quad\text{or}\quad 3(1 - \cos^2\theta) + \cos^2\theta = 5\cos\theta$$

or $\quad 2\cos^2\theta + 5\cos\theta - 3 = 0$

It is a quadratic in $\cos\theta$. Solving for $\cos\theta$,

$$\cos\theta = \frac{-5 \pm \sqrt{(5)^2 - 4(2)(-3)}}{2(2)} = \frac{-5 \pm 7}{4} = \frac{1}{2} \text{ or } -3$$

But $-1 \le \cos\theta \le 1$, therefore, $\cos\theta = -3$ is not possible.

$$\therefore\quad \cos\theta = \frac{1}{2} \Rightarrow \theta = 60^\circ$$

Example 10. *If* $\cos\theta = \frac{-3}{5}$ and $\pi < \theta < \frac{3\pi}{2}$, *find the values of remaining trigonometric ratios and hence evaluate* $\frac{\operatorname{cosec}\theta + \cot\theta}{\sec\theta - \tan\theta}$.

Solution: θ lies in the third quadrant, therefore, $\sin\theta < 0$, $\cos\theta < 0$ and $\tan\theta > 0$

We have

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \left(\frac{-3}{5}\right)^2 = 1 - \frac{9}{25} = \frac{16}{25}$$

$\therefore \quad \sin\theta = \pm\frac{4}{5}$, but $\sin\theta$ is $-ve$ $\quad\therefore\quad \sin\theta = -\frac{4}{5}$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{\left(\frac{-3}{5}\right)} = -\frac{5}{3}$$

$$\text{cosec}\,\theta = \frac{1}{\sin\theta} = \frac{1}{\frac{-4}{5}} = -\frac{5}{4}$$

$$\tan\theta = \frac{\sin 6}{\cos\theta} = \frac{-4/5}{-3/5} = \frac{4}{3}$$

$$\therefore \quad \cot\theta = \frac{1}{\tan\theta} = \frac{1}{\left(\frac{4}{3}\right)} = \frac{3}{4}$$

Now $\quad \dfrac{\text{cosec}\,\theta + \cot\theta}{\sec\theta - \tan\theta} = \dfrac{-5/4 + 3/4}{-5/3 - 4/3} = \dfrac{-2/4}{-9/3} = \dfrac{2}{4} \times \dfrac{3}{9} = \dfrac{1}{6}$

EXERCISE 3.2

LEVEL OF DIFFICULTY A

1. Find the values of the other five trigonometric functions in each of the following problems:

   (i) $\sin\theta = -\frac{3}{4}$, θ lies in third quadrant; (ii) $\tan\theta = 1/\sqrt{5}$, θ lies in first quadrant;

   (iii) $\cot\theta = \sqrt{2}/3$, θ lies in third quadrant; (iv) $\sec\theta = 2$, θ lies in fourth quadrant;

   (v) $\text{coesc}\,\theta = -2/\sqrt{3}$, θ lies in fourth quadrant;

   (vi) $\cos\theta = -\frac{1}{2}$, θ lies in third quadrant; (vii) $\sin\theta = \frac{3}{5}$, θ lies in second quadrant;

   (viii) $\cot\theta = \frac{3}{4}$, θ lies in third quadrant; (ix) $\sec\theta = \frac{13}{5}$, θ lies in fourth quadrant;

   (x) $\tan\theta = -\frac{5}{12}$, θ lies in the second quadrant.

2. Given $\cot\theta = 12/5$ and θ lies in third quadrant, find the values of other five trigonometric functions.

3. If $\cos\theta = -1/2$ and $\pi < \theta < 3\pi/2$, find the value of $4\tan^2\theta - 3\cos^2\theta$.

4. If $\sec\theta = \sqrt{2}$ and $3\pi/2 < \theta < 2\pi$, find the value of $\dfrac{1 + \tan\theta + \sec\theta}{1 + \cot\theta - \text{cosec}\,\theta}$.

5. If $\sin\theta = 7/25$ and θ lies in the second quadrant, find $\tan\theta$ and $\sec\theta$.

6. If $\sin\theta + \cos\theta = 0$ and θ lies in the fourth quadrant, find $\sin\theta$ and $\cos\theta$.

7. If $\sin\alpha = 12/13$, $\pi/2 < \alpha < \pi$ and $\sec\beta = 5/3$, $3\pi/2 < \beta < 2\pi$, find the value of $5\tan\alpha - 12\cot\beta$.

8. If $\sin\theta\sec\theta = -1$ and θ lies in the second quadrant, find the values of $\sin\theta$ and $\sec\theta$.

9. If $\cos\theta \operatorname{cosec}\theta = -1$ and θ lies in the fourth quadrant, find the values of $\cos\theta$ and $\operatorname{cosec}\theta$.

10. If A lies in the fourth quadrant and $\cos A = 5/13$, find the value of $\dfrac{13\sin A + 5\sec A}{5\tan A + 6\operatorname{cosec} A}$.

11. If $\sin\theta = \dfrac{21}{29}$, $0 < \theta < \dfrac{\pi}{2}$, show that $\sec\theta + \tan\theta = 2\dfrac{1}{2}$.

12. If θ lies in the fourth quadrant and $\cos\theta = \dfrac{3}{5}$, find the value of

$$\frac{5\sin\theta + 3\sec\theta - 3\tan\theta}{4\cot\theta + 3\operatorname{cosec}\theta + 5\cos\theta}.$$

13. If $\sin\alpha = \dfrac{3}{5}$, $\tan\beta = \dfrac{1}{2}$ and $\dfrac{\pi}{2} < \alpha < \pi < \beta < \dfrac{3\pi}{2}$, find the value of $8\tan\alpha - \sqrt{5}\sec\beta$.

14. Prove that $\dfrac{4}{3}\cot^2 30° + 3\sin^2 60° - 2\operatorname{cosec}^2 60° - \dfrac{4}{3}\tan^2 30° = 3\dfrac{1}{3}$.

15. Show that $\cos^2\dfrac{\pi}{4}\sec^2\dfrac{\pi}{3}\cos\dfrac{\pi}{2} - 15\sin^2\dfrac{\pi}{2}\cos\dfrac{\pi}{4} - 4\cos\dfrac{\pi}{6}\cos\dfrac{\pi}{4}\cos\dfrac{\pi}{3} = -\dfrac{15+\sqrt{3}}{\sqrt{2}}$.

16. Prove that
(i) $\cos^2 30°$, $\cos^2 45°$, $\cos^2 60°$ are in A.P. (ii) $\cot^2 30°$, $\cot^2 45°$, $\cot^2 60°$ are in G.P.

LEVEL OF DIFFICULTY B

17. Given angle C of a triangle ABC to be obtuse, find all the angles when

$$\sin(A+B) = \frac{\sqrt{3}}{2} \text{ and } \cos(A-B) = \frac{1}{\sqrt{2}}$$

18. If $\sin(A+B+C) = 1$, $\sin(A-B) = 1/2$ and $\cos(A+C) = 1/2$, find the values of A, B and C.

19. If $2\sin^2\theta - 5\sin\theta + 2 = 0$, $0 < \theta < \pi/2$, find the value of θ.

20. If $3\sec^4\theta + 8 = 10\sec^2\theta$, $0 < \theta < \pi/2$, find the values of θ.

21. Find the angles of a triangle, given that angle A is obtuse and

$$\sec(B+C) = \operatorname{cosec}(B-C) = 2$$

22. If $\dfrac{\pi}{2} < \alpha < \pi$, find the value of the expression

$$\sqrt{\frac{1-\sin\alpha}{1+\sin\alpha}} + \sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}}$$

23. If $\dfrac{\cos A}{3} = \dfrac{\cos B}{4} = \dfrac{1}{5}$, $-\dfrac{\pi}{2} < A < 0$, $-\dfrac{\pi}{2} < B < 0$, then find the value of $2\sin A + 4\sin B$.

Answers

1. (i) $\cos\theta = \dfrac{-\sqrt{7}}{4}$, $\tan\theta = \dfrac{3}{\sqrt{7}}$, $\cot\theta = \dfrac{\sqrt{7}}{3}$, $\sec\theta = \dfrac{-4}{\sqrt{7}}$, $\operatorname{cosec}\theta = \dfrac{-4}{3}$;

(ii) $\cot\theta = \sqrt{5}$, $\sec\theta = \sqrt{\dfrac{6}{5}}$, $\operatorname{cosec}\theta = \sqrt{6}$, $\sin\theta = \dfrac{1}{\sqrt{6}}$, $\cos\theta = \dfrac{\sqrt{5}}{\sqrt{6}}$;

(iii) $\tan\theta = \dfrac{3}{\sqrt{2}}$, $\sec\theta = -\sqrt{\dfrac{11}{2}}$, $\operatorname{cosec}\theta = \dfrac{-\sqrt{11}}{3}$, $\sin\theta = -\dfrac{3}{\sqrt{11}}$, $\cos\theta = -\sqrt{\dfrac{2}{11}}$;

(iv) $\operatorname{cosec}\theta = -\dfrac{2}{\sqrt{3}}$, $\cos\theta = \dfrac{1}{2}$, $\sin\theta = -\dfrac{\sqrt{3}}{2}$, $\tan\theta = -\sqrt{3}$, $\cot\theta = -\dfrac{1}{\sqrt{3}}$;

(v) $\sin\theta = -\frac{\sqrt{3}}{2}$, $\cos\theta = \frac{1}{2}$, $\cot\theta = -\frac{1}{\sqrt{3}}$, $\tan\theta = -\sqrt{3}$, $\sec\theta = 2$;

(vi) $\sin\theta = -\frac{\sqrt{3}}{2}$, $\tan\theta = \sqrt{3}$, $\cot\theta = \frac{1}{\sqrt{3}}$, $\sec\theta = -2$, $\operatorname{cosec}\theta = -\frac{2}{\sqrt{3}}$;

(vii) $\cos\theta = -\frac{4}{5}$, $\tan\theta = -\frac{3}{4}$, $\cot\theta = -\frac{4}{3}$, $\sec\theta = -\frac{5}{4}$, $\operatorname{cosec}\theta = \frac{5}{3}$;

(viii) $\sin\theta = -\frac{4}{5}$, $\cos\theta = -\frac{3}{5}$, $\sec\theta = -\frac{5}{3}$, $\tan\theta = \frac{4}{3}$, $\operatorname{cosec}\theta = -\frac{5}{4}$;

(ix) $\sin\theta = -\frac{12}{13}$, $\cot\theta = -\frac{5}{12}$, $\cos\theta = \frac{5}{3}$, $\tan\theta = -\frac{12}{5}$, $\operatorname{cosec}\theta = -\frac{13}{12}$;

(x) $\sin\theta = \frac{5}{13}$, $\cos\theta = -\frac{12}{13}$, $\cot\theta = -\frac{12}{5}$, $\operatorname{cosec}\theta = \frac{13}{5}$, $\sec\theta = -\frac{13}{12}$.

2. $\tan\theta = \frac{5}{12}$, $\sin\theta = -\frac{5}{13}$, $\cos\theta = -\frac{5}{13}$, $\sec\theta = -\frac{13}{12}$, $\operatorname{cosec}\theta = -\frac{13}{5}$.

3. 45/4. **4.** 1. **5.** $-\frac{7}{24}, -\frac{25}{24}$. **6.** $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$. **7.** –3.

8. $\frac{1}{\sqrt{2}}, -\sqrt{2}$. **9.** $\frac{1}{\sqrt{2}}, -\sqrt{2}$. **10.** $-\frac{2}{37}$. **12.** –4/3. **13.** –7/2.

17. $52\frac{1}{2}^\circ$, $7\frac{1}{2}^\circ$, 120°. **18.** $A = 60°$, $B = 30°$, $C = 0°$. **19.** $\frac{\pi}{6}$.

20. 30°, 45°. (i) 120°, 45°, 15°. **22.** $-\frac{2}{\cos\alpha}$. **23.** –4.

HINTS AND SOLUTIONS

6. We have

$$\sin\theta + \cos\theta = 0 \text{ or } \sin\theta = -\cos\theta \quad ...(1)$$

$$\Rightarrow \quad \frac{\sin\theta}{\cos\theta} = -1 \text{ i.e., } \tan\theta = -1$$

$$\sec^2\theta = 1 + \tan^2\theta = 1 + (-1)^2 = 1 + 1 = 2$$

$$\therefore \quad \sec\theta = \pm\sqrt{2} \text{ and } \cos\theta = \frac{1}{\sec\theta} = \pm\frac{1}{\sqrt{2}}$$

But $\cos\theta$ is $+ve$ in the fourth quadrant

$$\therefore \quad \cos\theta = \frac{1}{\sqrt{2}}$$

From (1),

$$\sin\theta = -\cos\theta = -\frac{1}{\sqrt{2}}$$

8. We have $\sin\theta\sec\theta = -1$

$$\therefore \quad \sin\theta = -\frac{1}{\cos\theta} = -1, \text{ i.e., } \tan\theta = -1$$

Now, $\sec^2\theta = 1 + \tan^2\theta = 1 + (-1)^2 = 1 + 1 = 2$

$$\therefore \quad \sec\theta = -\sqrt{2} \qquad (\because \theta \text{ is in II quadrant})$$

Given equation implies $\sin\theta = \frac{-1}{\sec\theta} = \frac{-1}{-\sqrt{2}} = \frac{1}{\sqrt{2}}$

$$\therefore \quad \sin\theta = 1/\sqrt{2} \text{ and } \sec\theta = -\sqrt{2}$$

10. $\cos A = \frac{5}{13}$, $\quad \therefore \quad \sec A = \frac{1}{\cos A} = \frac{13}{5}$

$$\sin^2 A = 1 - \cos^2 A = 1 - \frac{25}{169} = \frac{144}{169}$$

$\therefore \quad \sin A = \pm\frac{12}{13}$

$\because \quad A$ lies in the fourth quadrant

$\therefore \quad \sin A = -\frac{12}{13} \Rightarrow \operatorname{cosec} A = \frac{1}{\sin A} = -\frac{13}{12}$

$$\tan A = \frac{\sin A}{\cos A} = \frac{-12/13}{5/13} = -\frac{12}{5}$$

$$\therefore \quad \frac{13\sin A + 5\sec A}{5\tan A + 6\operatorname{cosec} A} = \frac{13\left(-\frac{12}{13}\right)+5\left(\frac{13}{5}\right)}{5\left(-\frac{12}{5}\right)+6\left(-\frac{13}{12}\right)} = \frac{-12+13}{-12-\frac{13}{2}} = \frac{1}{-\frac{37}{2}} = -\frac{2}{37}.$$

17. In ΔABC, $A + B + C = 180°$. ...(1)

$\angle C$ is obtuse $\Rightarrow \angle s\ A$ and B are positive acute angles.

$\sin(A + B) = \frac{\sqrt{3}}{2} = \sin 60° \Rightarrow A + B = 60°$ ...(2)

$\cos(A - B) = \frac{1}{\sqrt{2}} = \cos 45° \Rightarrow A - B = 45°$ ...(3)

Adding (2) and (3), $\quad 2A = 105° \quad \therefore \quad A = 52\frac{1°}{2}$

Subtracting (3) from (2), $2B = 15° \quad \therefore \quad B = 7\frac{1°}{2}$

Subtracting (2) from (1), $C = 120°$.

18. $\sin(A + B + C) = 1 \Rightarrow A + B + C = 90°$ ...(1)

$\sin(A - B) = \frac{1}{2} \Rightarrow A - B = 30°$ ...(2)

$\cos(A + C) = \frac{1}{2} \Rightarrow A + C = 60°$ ...(3)

Subtracting (3) from (1), we get $B = 30°$

$\therefore$ From (2), $A - 30° = 30° \Rightarrow A = 60°$

From (3), $60° + C = 60° \Rightarrow C = 0°$

Hence $A = 60°$, $B = 30°$ and $C = 0°$.

20. $3\sec^4\theta - 10\sec^2\theta + 8 = 0 \Rightarrow (3\sec^2\theta - 4)(\sec^2\theta - 2) = 0$

$$\Rightarrow \sec\theta = 2/\sqrt{3} \text{ or } \sqrt{2} \text{ since } \sec\theta > 0.$$

Therefore, $\theta = 30°$ or $45°$. $[\because \sec 30° = 2/\sqrt{3}, \sec 45° = \sqrt{2}]$

21. Since A, B, C, are angles of triangle, $\therefore\ A + B + C = 180°$ ...(1)

Since A is obtuse, $\therefore\ B$ and C are both acute angles.

Now, $\sec(B + C) = 2 = \sec 60° \quad \therefore \quad B + C = 60°$ ...(2)

$\operatorname{cosec}(B - C) = 2 = \operatorname{cosec} 30° \quad \therefore \quad B - C = 30°$ ...(3)

From (2) and (3), $B = 45°$, $C = 15°$

From (1), we get $A = 120°$.

22. Given, $\frac{\pi}{2} < \alpha < \pi$

$$\sqrt{\frac{1-\sin\alpha}{1+\sin\alpha}}+\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}} = \sqrt{\frac{(1-\sin\alpha)^2}{\cos^2\alpha}}+\sqrt{\frac{(1+\sin\alpha)^2}{\cos^2\alpha}}$$

$$= \left|\frac{1-\sin\alpha}{\cos\alpha}\right|+\left|\frac{1+\sin\alpha}{\cos\alpha}\right| = -\frac{1-\sin\alpha}{\cos\alpha}-\frac{1+\sin\alpha}{\cos\alpha}$$

$$= -\frac{2}{\cos\alpha} \qquad [\because \cos\alpha < 0, \sin\alpha > 0]$$

23. Given $\cos A = \frac{3}{5}$, $\cos B = \frac{4}{5}$

$\therefore \quad \sin A = -\frac{4}{5}$, $\sin B = -\frac{3}{5}$ [$\because$ A, B lie in 4th quadrant]

Now, $\quad 2\sin A + 4\sin B = -\frac{8}{5} - \frac{12}{5} = -4.$

TRIGONOMETRIC RATIOS OF ALLIED ANGLES

Two angles are called *allied angles* if either

(i) their sum is zero or

(ii) their sum or difference is a multiple of right angle.

(i) *t*-ratios of $(-\theta)$ in terms of those of θ (assumed to be +ve acute angle):

ΔOMP is a right angled triangle of acute angle θ and $\Delta OMP'$ is a rt. angled triangle with angle $(-\theta)$. Therefore, ΔOMP and OMP' are similar. Refer to Fig. 3.28. But $MP' = -MP$, $OP = OP'$

In $\Delta\, OMP'$,

$$\sin(-\theta) = \frac{MP'}{OP'} = -\frac{MP}{OP} = -\sin\theta$$

$$\cos(-\theta) = \frac{OM}{OP'} = \frac{OM}{OP} = \cos\theta$$

$$\tan(-\theta) = \frac{MP'}{OM} = -\frac{MP}{OM} = -\tan\theta$$

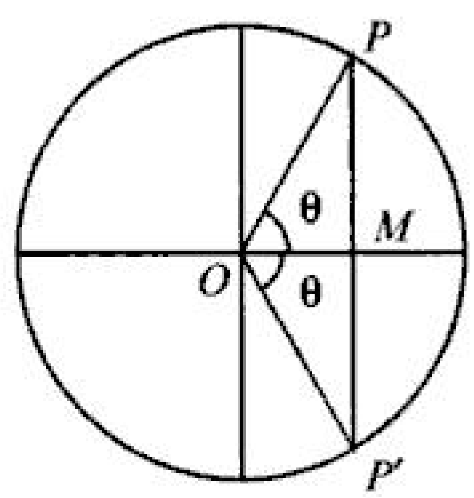

Fig. 3.28

Similarly,

$$\text{cosec}(-\theta) = -\text{cosec}\,\theta$$
$$\sec(-\theta) = \sec\theta$$
$$\cot(-\theta) = -\cot\theta$$

(ii) *t*-ratios of $(90° + \theta)$ in terms of θ:

ΔOMP is right angled with angle $MOP = \theta$ and $\Delta OM'P'$ is right angled with angle $MOP' = (90° + \theta)$. Δ's OMP and $OM'P'$ are similar. Refer to Fig. 3.29.

$\therefore \quad M'P' = OM$ and $OM' = -MP$ $OP' = OP$

In $\Delta OM'P'$,

$$\sin(90° + \theta) = \frac{M'P'}{OP'} = \frac{OM}{OP} = \cos\theta$$

$$\cos(90° + \theta) = \frac{OM'}{OP'} = -\frac{MP}{OP} = -\sin\theta$$

$$\tan(90° + \theta) = \frac{M'P'}{OM'} = -\frac{OM}{MP} = -\cot\theta$$

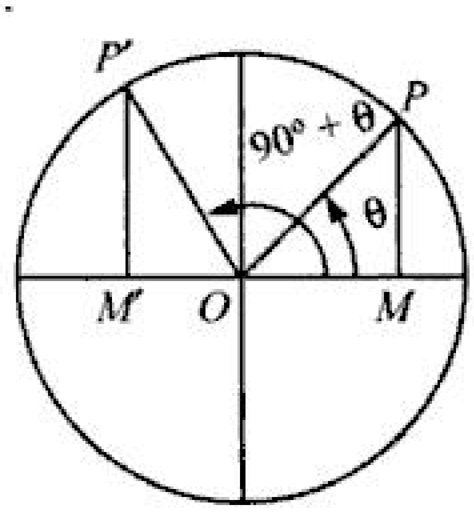

Fig. 3.29

Similarly,

$$\text{cosec}\,(90^\circ + \theta) = \sec\theta$$
$$\sec\,(90^\circ + \theta) = -\text{cosec}\,\theta$$
$$\cot\,(90^\circ + \theta) = -\tan\theta.$$

(iii) *t*-ratios of $(90^\circ - \theta)$ in terms of θ:

Changing θ into $-\theta$ in $(90^\circ + \theta)$, we get

$$\sin(90^\circ - \theta) = \sin[90^\circ + (-\theta)] = \cos(-\theta) = \cos\theta \qquad \text{[Using results (i) and (ii)]}$$
$$\cos(90^\circ - \theta) = \cos[90^\circ + (-\theta)] = -\sin(-\theta) = \sin\theta$$
$$\tan(90^\circ - \theta) = \tan[90^\circ + (-\theta)] = -\cot(-\theta) = \cot\theta$$

Similarly, we may get remaining *t*-ratios.

(iv) *t*-ratios of $(180^\circ + \theta)$ in terms of θ:

$$\sin(180^\circ + \theta) = \sin[90^\circ + (90^\circ + \theta)] = \cos(90^\circ + \theta) = -\sin\theta$$
$$\cos(180^\circ + \theta) = \cos[90^\circ + (90^\circ + \theta)] = -\sin(90^\circ + \theta) = -\cos\theta$$
$$\tan(180^\circ + \theta) = \tan[90^\circ + (90^\circ + \theta)] = -\cot(90^\circ + \theta) = \tan\theta$$

Similarly, we may get other *t*-ratios.

(v) *t*-ratios of $(180^\circ + \theta)$ in terms of θ:

Changing θ into $(-\theta)$ in result (iv), we get

$$\sin(180^\circ - \theta) = \sin\theta$$
$$\cos(180^\circ - \theta) = -\cos\theta$$
$$\tan(180^\circ - \theta) = -\tan\theta$$

Similarly, we may get other *t*-ratios.

(vi) *t*-ratios of $(270^\circ + \theta)$ in terms of θ:

$$\sin(270^\circ + \theta) = \sin[180^\circ + (90^\circ + \theta)] = -\sin(90^\circ + \theta) = -\cos\theta$$
$$\cos(270^\circ + \theta) = \cos[180^\circ + (90^\circ + \theta)] = -\cos(90^\circ + \theta) = -\sin\theta$$
$$\tan(270^\circ + \theta) = \tan[180^\circ + (90^\circ + \theta)] = \tan(90^\circ + \theta) = -\cot\theta$$

Similarly, we get other *t*-ratios.

(vii) *t*-ratios of $(270^\circ - \theta)$ in terms of θ:

Changing θ into $(-\theta)$ in result (vi), we get

$$\sin(270^\circ - \theta) = -\cos\theta$$
$$\cos(270^\circ - \theta) = -\sin\theta$$
$$\tan(270^\circ - \theta) = \cot\theta$$

Similarly, we may get other *t*-ratios.

(viii) *t*-ratios of $(360^\circ - \theta)$ in terms of θ:

We get the same values as in result (i), hence

$$\sin(360^\circ - \theta) = -\sin\theta$$
$$\cos(360^\circ - \theta) = \cos\theta$$
$$\tan(360^\circ - \theta) = -\tan\theta$$

(ix) *t*-ratios of (360° + θ) in terms of θ:

After one complete rotation, we have the same ΔOMP, i.e., Δ's for θ and (360° + θ) is the same (*OMP*), and hence we get the same ratios as for θ, i.e.,

$$\sin(360° + \theta) = \sin\theta$$
$$\cos(360° + \theta) = \cos\theta$$
$$\tan(360° + \theta) = \tan\theta$$

Similarly, we may get other *t*-ratios.

(x) After *n* complete rotations, we have the same ΔOMP as for θ and hence we get the same ratios. That is,

$$\sin(2n\pi + \theta) = \sin\theta$$
$$\cos(2n\pi + \theta) = \cos\theta$$
$$\tan(2n\pi + \theta) = \tan\theta.$$

Remark. All the trigonometrical ratios discussed earlier can be presented in the form of Table 3.1.

Table 3.1 Allied Angles

t-ratios	$-\theta$	$90° - \theta$	$90° + \theta$	$180° - \theta$	$180° + \theta$	$270° - \theta$	$270° + \theta$	$360° - \theta$	$360° + \theta$
$\sin\theta$	$-\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$
$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$	$\cos\theta$	$\cos\theta$
$\tan\theta$	$-\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$
$\operatorname{cosec}\theta$	$-1 \times \operatorname{cosec}\theta$	$\sec\theta$	$\sec\theta$	$\operatorname{cosec}\theta$	$-1 \times \operatorname{cosec}\theta$	$-\sec\theta$	$-\sec\theta$	$-1 \times \operatorname{cosec}\theta$	$\operatorname{cosec}\theta$
$\sec\theta$	$\sec\theta$	$\operatorname{cosec}\theta$	$-1 \times \operatorname{cosec}\theta$	$-\sec\theta$	$-\sec\theta$	$-1 \times \operatorname{cosec}\theta$	$\operatorname{cosec}\theta$	$\sec\theta$	$\sec\theta$
$\cot\theta$	$-\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$

RULE TO REMEMBER THE T-RATIOS OF ALLIED ANGLES

(i) For angles with *X'OX*, i.e., $-\theta$, $180° - \theta$, $180° + \theta$, $360° - \theta$ and $360° + \theta$, we write the same *t*-ratio with proper sign. For example,

$$\sin(180° + \theta) = -\sin\theta$$
$$\cos(360° - \theta) = \cos\theta$$
$$\tan(180° - \theta) = -\tan\theta$$

(ii) For angle with *Y'OY*, i.e., $90° - \theta$, $90° + \theta$, $270° - \theta$, $270° + \theta$, we write

sin → cos
cos → sin
tan → cot etc. with proper sign

e.g.

$$\tan(90° + \theta) = -\cot\theta$$
$$\sin(270° + \theta) = -\cos\theta$$
$$\cos(270° - \theta) = -\sin\theta, \text{ etc.}$$

Note: By proper sign, we mean the quadrant in which the angle lies and remembering that in

i quadrant → all are +ve
ii quadrant → sin is +ve
iii quadrant → tan is +ve
iv quadrant → cos is +ve

Illustrations. (i) $\sin(-\theta) = -\sin\theta$

$\because$ $-\theta$ lies in the fourth quadrant in which sine is –ve.

Also, by (i) sine remains sine.

(ii) $\cos(90° - \theta) = \sin\theta$

$\because$ $90° - \theta$ lies in the first quadrant in which cosine is +ve.

Also, by (ii) cosine changes into sine. (remove 'co')

(iii) $\tan(90° + \theta) = -\cot\theta$

$\because$ $90° + \theta$ lies in the second quadrant in which tangent is –ve.

Also, by (ii) tangent changes into cotangent. (add 'co')

(iv) $\operatorname{cosec}(180° - \theta) = \operatorname{cosec}\theta$

$\because$ $180° - \theta$ lies in the second quadrant in which cosecant is +ve.

Also, by (i) cosecant remains cosecant.

Example 1. *Find the values of:*

(i) $\cos 135°$; (ii) $\sec 120°$; (iii) $\operatorname{cosec} 150°$; (iv) $\cos 225°$; (v) $\tan 120°$; (vi) $\sin 315°$; (vii) $\cos(-300)°$; (viii) $\cot 330°$; (ix) $\cos 405°$; (x) $\sin\left(-\frac{11\pi}{3}\right)$; (xi) $\sin\frac{31\pi}{3}$; (xii) $\cos(-1710°)$.

Solution:

(i) $\cos 135° = \cos(90° + 45°) = -\sin 45° = -\frac{1}{\sqrt{2}}$

(ii) $\sec 120° = \sec(90° + 30°) = -\operatorname{cosec} 30° = -2$

(iii) $\operatorname{cosec} 150° = \operatorname{cosec}(90° + 60°) = \sec 60° = 2$

(iv) $\cos 225° = \cos(180° + 45°) = -\cos 45° = -\frac{1}{\sqrt{2}}$

(v) $\tan 120° = \tan(180° - 60°) = -\tan 60° = -\sqrt{3}$

(vi) $\sin 315° = \sin(360° - 45°) = -\sin 45° = -\frac{1}{\sqrt{2}}$

(vii) $\cos(-300°) = \cos 300° = \cos(360° - 60°) = \cos 60° = \frac{1}{2}$

(viii) $\cot 330° = \cot(270° + 60°) = -\tan 60° = -\sqrt{3}$

(ix) $\cos 405° = \cos(360° + 45°) = \cos 45° = \frac{1}{\sqrt{2}}$

(x) $\sin\left(-\frac{11\pi}{3}\right) = -\sin\frac{11\pi}{3} = -\sin\left(4\pi - \frac{\pi}{3}\right) = -\left(-\sin\frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

(xi) $\sin\frac{31\pi}{3} = \sin\left(10\pi + \frac{\pi}{3}\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

(xii) $\cos(-1710°) = \cos(-1710° + 5 \times 360°) = \cos(-1710° + 1800°) = \cos 90° = 0$

Example 2. *Find the values of:*

(i) $\cos 1050°$; (ii) $\sin(-2025°)$; (iii) $\sec 4620°$; (iv) $\tan(-945°)$;
(v) $\text{cosec}(-1170°)$; (vi) $\cot 1215°$; (vii) $\cos(-960°)$; (viii) $\sin 1230°$.

Solution: (i) $\cos 1050° = \cos(2 \times 360° + 330°)$

$$= \cos 330° = \cos(360° - 30°) = \cos 30° = \frac{\sqrt{3}}{2};$$

(ii) $\sin(-2025°) = -\sin 2025° = -\sin(5 \times 360° + 225°) = -\sin 225°$

$$= -\sin(180° + 45°) = -(-\sin 45°) = \sin 45° = \frac{1}{\sqrt{2}};$$

(iii) $\sec 4620° = \sec(12 \times 360° + 300°) = \sec 300° = \sec(360° - 60°) = \sec 60° = 2;$

(iv) $\tan(-945°) = -\tan 945° = -\tan(2 \times 360° + 225°)$

$$= -\tan 225° = -\tan(180° + 45°) = -\tan 45° = -1;$$

(v) $\text{cosec}(-1170°) = -\text{cosec } 1170° = -\text{cosec}(3 \times 360° + 90°) = -\text{cosec } 90° = -1;$

(vi) $\cot 1215° = \cot(3 \times 360° + 135°) = \cot 135° = \cot(180° - 45°) = -\cot 45° = -1;$

(vii) $\cos(-960°) = \cos 960° = \cos(2 \times 360° + 240°) = \cos 240°$

$$= \cos(270° - 30°) = -\sin 30° = -\frac{1}{2};$$

(viii) $\sin 1230° = \sin(3 \times 360° + 150°) = \sin 150° = \sin(180° - 30°) = \sin 30° = \frac{1}{2}.$

Example 3. *Prove that* $\tan 315° \cot(-405°) + \cot 495° \tan(-585°) = 2.$

Solution: $\tan 315° = \tan(360° - 45°) = -\tan 45° = -1$

$$\cot(-405°) = -\cot 405° = -\cot(360° + 45°) = -\cot 45° = -1$$

$$\cot 495° = \cot(360° + 135°) = \cot 135° = \cot(180° - 45°) = -\cot 45° = -1$$

$$\tan(-585°) = -\tan(585°) = -\tan(360° + 225°) = -\tan 225° = -\tan(180° + 45°)$$

$$= -\tan 45° = -1$$

$\therefore$ L.H.S. $= (-1)(-1) + (-1)(-1) = 1 + 1 = 2 =$ R.H.S.

Example 4. *Prove that* $\sin(-690°)\cos(-300°) + \cos(-750°)\sin(-240°) = 1.$

Solution: $\sin(-690°) = \sin(-2 \times 360 + 30)° = \sin 30° = \frac{1}{2},$

$$\cos(-300°) = \cos(-360° + 60°) = \cos 60° = \frac{1}{2},$$

$$\cos(-750°) = \cos(-2 \times 360° - 30)° = \cos(-30°) = \cos 30° = \frac{\sqrt{3}}{2},$$

$$\sin(-240°) = -\sin 240° = -\sin(180° + 60°) = -(-\sin 60°) = \sin 60° = \frac{\sqrt{3}}{2}$$

$$\therefore \sin(-690°)\cos(-300°) + \cos(-750°)\sin(-240°) = \frac{1}{2} \cdot \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = \frac{1}{4} + \frac{3}{4} = 1.$$

Example 5. *Simplify*

$$\frac{\tan(90° + \theta)\sin(180° + \theta)\sec(270° + \theta)}{\cos(270° - \theta)\text{cosec}(180° - \theta)\cot(360° - \theta)}$$

Solution: The given expression is

$$\frac{\tan(90°+\theta)\sin(180°+\theta)\sec(270°+\theta)}{\cos(270°-\theta)\text{cosec}(180°-\theta)\cot(360°-\theta)} = \frac{(-\cot\theta)(-\sin\theta)(\text{cosec}\,\theta)}{(-\sin\theta)(\text{cosec}\,\theta)(-\cot\theta)} = 1$$

Example 6. *Prove that*

(i) $\cos 24° + \cos 55° + \cos 125° + \cos 204° + \cos 300° = \frac{1}{2}$

(ii) $\cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8} + \cos^2\frac{5\pi}{8} + \cos^2\frac{7\pi}{8} = 2.$

Solution: (i) L.H.S. $= \cos 24° + \cos 55° + \cos 125° + \cos 204° + \cos 300°$

$$= \cos 24° + \cos 55° + \cos(180° - 55°) + \cos(180° + 24°) + \cos(360° - 60°)$$

$$= \cos 24° + \cos 55° - \cos 55° - \cos 24° + \cos 60°$$

$$= \cos 60° = \frac{1}{2} = \text{R.H.S.}$$

(ii) $\cos\frac{5\pi}{8} = \sin\left(\frac{\pi}{2} - \frac{5\pi}{8}\right)$ $\left[\because \cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)\right]$

$$= \sin\left(\frac{4\pi - 5\pi}{8}\right) = \sin\left(-\frac{\pi}{8}\right) = -\sin\frac{\pi}{8}$$ $[\because \cos(-\theta) = \sin\theta]$

$$\cos\frac{7\pi}{8} = \sin\left(\frac{\pi}{2} - \frac{7\pi}{8}\right) = \sin\left(\frac{4\pi - 7\pi}{8}\right) = \sin\left(-\frac{3\pi}{8}\right) = -\sin\frac{3\pi}{8}$$

$$\text{L.H.S.} = \cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8} + \cos^2\frac{5\pi}{8} + \cos^2\frac{7\pi}{8}$$

$$= \cos^2\frac{\pi}{8} + \cos^2\frac{3\pi}{8} + \sin^2\frac{\pi}{8} + \sin^2\frac{3\pi}{8}$$

$$= \left(\cos^2\frac{\pi}{8} + \sin^2\frac{\pi}{8}\right) + \left(\cos^2\frac{3\pi}{8} + \sin^2\frac{3\pi}{8}\right) = 1 + 1 = 2 = \text{R.H.S.}$$

Example 7. *Find x from the equation*

$$\sec(90° + A) + x\sin A\tan(90° + A) = \cos(90° + A)$$

Solution: $\sec(90° + A) = -\text{cosec}\,A$; $\tan(90° + A) = -\cot A$

$\cos(90° + A) = -\sin A$

$\therefore$ Given equation becomes

$$-\text{cosec}\,A + x\sin A(-\cot A) = -\sin A \Rightarrow x\sin A\left(\frac{-\cos A}{\sin A}\right) = \frac{1}{\sin A} - \sin A$$

$$\Rightarrow -x\cos A = \frac{1 - \sin^2 A}{\sin A} = \frac{\cos^2 A}{\sin A}$$

$$\Rightarrow x = -\frac{\cos^2 A}{\cos A\sin A} = -\cot A$$

Example 8. *Show that in cyclic quadrilateral ABCD*

(i) $\tan A + \tan B + \tan C + \tan D = 0$;

(ii) $\cos(180° - A) + \cos(180° + B) + \cos(180° + C) - \sin(90° + D) = 0.$

Solution: In a cyclic quadrilateral $ABCD$,

$$A + C = 180° \quad \text{and} \quad B + D = 180° \qquad ...(1)$$

(i) $\tan A + \tan B + \tan C + \tan D$

$= \tan A + \tan B + \tan(180° - A) + \tan(180° - B)$ [Using (1)]

$= \tan A + \tan B - \tan A - \tan B = 0$

(ii) $\cos(180° - A) + \cos(180° + B) + \cos(180° + C) - \sin(90° + D)$

$= -\cos A - \cos B - \cos C - \cos D$

$= -\cos A - \cos B - \cos(180° - A) - \cos(180° - B)$ [Using (1)]

$= -\cos A - \cos B + \cos A + \cos B$ [$\because \cos(180° - \theta) = -\cos\theta$]

$= 0$

Example 9. *Find all the positive values of x less than 360°, which satisfy 3 $\tan^2 x = 1$.*

Solution: $3\tan^2 x = 1 \Rightarrow \tan^2 x = \dfrac{1}{3} \Rightarrow \tan x = \pm\dfrac{1}{\sqrt{3}}$

Case I When $\tan x = \dfrac{1}{\sqrt{3}}$, which is $+ve$, x lies in first or third quadrant.

$\therefore \quad \tan x = \tan 30°$ or $\tan(180° + 30°)$ [$\because \tan\theta = \tan(180° + \theta)$]

$= \tan 30°$ or $\tan 210°$

$\Rightarrow x = 30°, 210°$

Case II When $\tan x = -\dfrac{1}{\sqrt{3}}$, which is $-ve$, x lies in 2nd or 4th quadrant.

$\therefore \quad \tan x = -\tan 30° = \tan(180° - 30°)$ or $\tan(360° - 30°) = \tan 150°$ or $\tan 330°$

$\Rightarrow x = 150°$ or $330°$

Hence $x = 30°, 150°, 210°, 330°$ are the required value.

Example 10. *Find all the positive values of x less than 2π, which satisfy* $\sin^2 x = \dfrac{1}{4}$.

Solution: $\sin^2 x = \dfrac{1}{4} \Rightarrow \sin x = \pm\dfrac{1}{2}$

Case I When $\sin x = \dfrac{1}{2}$ Here x lies in 1st quadrant or 2nd quadrant.

$\therefore \quad \sin x = \dfrac{1}{2} = \sin 30° = \sin(180° - 30°)$ or $\sin x = \sin 30° = \sin 150°$

Hence $\quad x = 30°, 150°$.

Case II When $\sin x = -\dfrac{1}{2}$. Here x lies in 3rd quadrant or 4th quadrant.

$\therefore \quad \sin x = -\dfrac{1}{2} = -\sin 30° = \sin(180° + 30°) = \sin(360° - 30°)$

or $\quad \sin x = \sin 210° = \sin 330°$

Hence $\quad x = 210°, 330° \quad \therefore \quad x = \dfrac{\pi}{6}, 5\dfrac{\pi}{6}, 7\dfrac{\pi}{8}, 11\dfrac{\pi}{6}$

Example 11. (a) *Prove that*

$$\left[1+\cot\alpha-\sec\left(\alpha+\frac{\pi}{2}\right)\right]\left[1+\cot\alpha+\sec\left(\alpha+\frac{\pi}{2}\right)\right]=2\cot\alpha$$

Solution: $\sec\left(\alpha+\frac{\pi}{2}\right) = -\text{cosec }\alpha$

$\therefore$ L.H.S. $= [(1 + \cot\alpha) + \text{cosec }\alpha]\,[(1 + \cot\alpha) - \text{cosec }\alpha]$

$= (1 + \cot\alpha)^2 - \text{cosec}^2\alpha = 1 + \cot^2\alpha + 2\cot\alpha - \text{cosec}^2\alpha$

$[\because 1 + \cot^2\alpha = \text{cosec}^2\alpha]$

$= \text{cosec}^2\alpha + 2\cot\alpha - \text{cosec}^2\alpha = 2\cot\alpha =$ R.H.S.

EXERCISE 3.3

LEVEL OF DIFFICULTY A

1. Find the values of:

(i) $\tan 135°$; (ii) $\cot 120°$; (iii) $\tan 240°$; (iv) $\sin 315°$; (v) $\sin\left(-\frac{5\pi}{3}\right)$;

(vi) $\tan\frac{11\pi}{6}$; (vii) $\sin(-330°)$; (viii) $\cos 495°$;

(ix) $\sin 765°$; (x) $\tan\frac{19\pi}{3}$; (xi) $\cot\left(-\frac{15\pi}{4}\right)$; (xii) $\tan\frac{19\pi}{3}$.

2. Find the values of:

(i) $\sec(-1680°)$; (ii) $\tan(-585°)$; (iii) $\cos 870°$;

(iv) $\sin 1230°$; (v) $\text{cosec}(-1200°)$; (vi) $\text{cosec}(-1560°)$.

3. Prove that:

(i) $\sin 420°\cos 390° + \cos(-660°)\sin(-330°) = 1$;

(ii) $\cos 510°\cos 330° + \sin 390°\cos 120° = -1$;

(iii) $\sin 600°\tan(-690°) + \sec 840°\cot(-945°) = \frac{3}{2}$;

(iv) $\cos 570°\sin 510° + \sin(-330°)\cos(-390°) = 0$;

(v) $\sin 150°\cos 120° + \cos 330°\sin 660° = -1$.

4. Prove that:

(i) $\sin^2 54° - \sin^2 72° = \sin^2 18° - \sin^2 36°$;

(ii) $\sin 75° - \sin 15° = \cos 105° + \cos 15°$;

(iii) $\cos 306° + \cos 234° + \cos 162° + \cos 18° = 0$;

(iv) $\sin^2\frac{\pi}{4}+\sin^2\frac{3\pi}{4}+\sin^2\frac{5\pi}{4}+\sin^2\frac{7\pi}{4} = 2$;

(v) $\sin^2\frac{\pi}{8}+\sin^2\frac{3\pi}{8}+\sin^2\frac{5\pi}{8}+\sin^2\frac{7\pi}{8} = 2$;

(vi) $\cos^3\frac{\pi}{8}+\cos^3\frac{3\pi}{8}+\cos^3\frac{5\pi}{8}+\cos^3\frac{7\pi}{8} = 0$.

5. Prove that:

(i) $\dfrac{\cos(90°+\theta)\sec(-\theta)\tan(180°-\theta)}{\sec(360°-\theta)\sin(180°+\theta)\cot(90°-\theta)} = -1;$

(ii) $\dfrac{\sin(-\theta)\tan(180°-\theta)\tan(190°+\theta)}{\cot(90°-\theta)\cos(360°+\theta)\sin(180°-\theta)} = -\operatorname{cosec}\theta;$

(iii) $\dfrac{\sin(-\theta)\tan(190°+\theta)\sin(180°+)\sec(270°+\theta)}{\sin(360°-\theta)\cos(270°-\theta)\operatorname{cosec}(180°-\theta)\cot(360°-\theta)} = 1;$

(iv) $\dfrac{\cos(2\pi+\theta)\operatorname{cosec}(4\pi+\theta)\tan\left(\dfrac{\pi}{2}+\theta\right)}{\sec\left(\dfrac{\pi}{2}+\theta\right)\cos\theta\cot(\pi+\theta)} = 1;$

(v) $\dfrac{\sin(180°+A)\sec(270+A)\tan(90°+A)}{\operatorname{cosec}(180°-A)\cos(270°+A)\cot(360°-A)} = 1.$

6. Find x from the following equations:

(i) $\operatorname{cosec}(90°+A) + x\cos A\cot(90°+A) = \sin(90°+A);$

(ii) $\cos(180°+\theta) + x\sin(90°+\theta)\cot(270°+\theta) = \operatorname{cosec}(270°+\theta);$

(iii) $x\cot(90°+A) + \tan(90°+A)\sin A + \operatorname{cosec}(90°+A) = 0.$

7. In any quadrilateral $ABCD$, prove that

(i) $\sin(A+B) + \sin(C+D) = 0;$ (ii) $\cos(A+B) = \cos(C+D).$

8. Find all the values of x between 0 and 2π which satisfy the following equations:

(i) $4\sin^2 x = 3;$ (ii) $3\tan^2 x = 1;$ (iii) $\sec x = -2;$ (iv) $\tan x = 1.$

9. (i) Show that $\dfrac{\sin 135° - \cos 120°}{\sin 135° + \cos 120°} = 3 + 2\sqrt{2}.$

(ii) Prove that:

(a) $\sin^2\dfrac{\pi}{6} + \cos^2\dfrac{\pi}{3} - \tan^2\dfrac{\pi}{4} = -\dfrac{1}{2};$

(b) $2\sin^2\dfrac{\pi}{6} + \operatorname{cosec}^2\dfrac{7\pi}{6}\cos^2\dfrac{\pi}{3} = \dfrac{3}{2};$

(c) $\cot^2\dfrac{\pi}{6} + \operatorname{cosec}\dfrac{5\pi}{6} + 3\tan^2\dfrac{\pi}{6} = 6;$

(d) $2\sin^2\dfrac{3\pi}{4} + 2\cos^2\dfrac{\pi}{4} + 2\sec^2\dfrac{\pi}{3} = 10.$

LEVEL OF DIFFICULTY B

10. Prove that $\sec\left(\dfrac{3\pi}{2}-\theta\right)\sec\left(\theta-\dfrac{5\pi}{2}\right)+\tan\left(\dfrac{5\pi}{2}+\theta\right)\tan\left(\theta-\dfrac{3\pi}{2}\right)=-1.$

11. Prove that in any triangle ABC

(i) $\sin(A+B) = \sin C;$ (ii) $\cos(A+B) + \cos C = 0;$

(iii) $\cos\dfrac{A+B}{2} = \sin\dfrac{C}{2};$ (iv) $\tan\dfrac{A+B}{2} = \cot\dfrac{C}{2}.$

12. If A, B, C, D be the angles of a cyclic quadrilateral, taken in order, prove that

(i) $\cos A + \cos B + \cos C + \cos D = 0$;

(ii) $\cos(180^\circ + A) + \cos(180^\circ + B) + \cos(180^\circ + C) - \sin(90^\circ + D) = 0$.

13. Show that $\sin[n\pi + (-1)^n x] = \sin x$ for every integer n.

14. Show that $\sin(n\pi + \theta) = (-1)^n \sin\theta$ for every integer.

15. Find the value of $\sin\left[n\pi + (-1)^n \frac{\pi}{6}\right]$, where n is any integer.

16. If $\tan 35^\circ = x$, prove that $\frac{\tan 145^\circ - \tan 125^\circ}{1 + \tan 145^\circ \tan 125^\circ} = \frac{1-x^2}{2x}$.

17. (i) Which is greater: $\sin 40^\circ$ or $\cos 40^\circ$?

(ii) If $\theta = -400^\circ$, determine the sign of $(\sin\theta + \cos\theta)$.

18. Find all the values of θ satisfying $0 < \theta < \pi$ and $\tan^2\theta + \cot^2\theta = 2$.

19. If $A = \frac{11\pi}{4}$, prove that $\sin^2 A - \cos^2 A + 2\tan A - \sec^2 A = -4$.

20. If $B + C = 60^\circ$, prove that $\sin(120^\circ - B) = \sin(120^\circ - C)$.

21. Find the value of $\prod_{k=0}^{6} \sin\frac{(2k+1)\pi}{14}$.

22. Show that $\cos^2\frac{\pi}{16} + \cos\frac{3\pi}{16} + \cos^2\frac{5\pi}{16} + \cos^2\frac{7\pi}{16} = 2$.

Answers

1. (i) -1; (ii) $-\frac{1}{\sqrt{3}}$; (iii) $\sqrt{3}$; (iv) $-\frac{1}{\sqrt{2}}$; (v) $\frac{\sqrt{3}}{2}$;

(vi) $-\frac{1}{\sqrt{3}}$; (vii) $\frac{1}{2}$; (viii) $-\frac{1}{\sqrt{2}}$; (ix) $\frac{1}{\sqrt{2}}$; (x) $\sqrt{3}$; (xi) 1; (xii) $\sqrt{3}$.

2. (i) -2; (ii) -1; (iii) $-\frac{\sqrt{3}}{2}$; (iv) $\frac{1}{2}$; (v) $-\frac{2}{\sqrt{3}}$; (vi) $\sqrt{2}$.

6. (i) $\tan A$; (ii) $\tan\theta$; (iii) $\sin A$.

8. (i) 60°, 120°, 240°, 300°; (ii) 30°, 150°, 210°, 330°; (iii) 120°, 240°; (iv) 45°, 225°.

15. $\frac{1}{2}$. **17.** cos 40°, greater than 0. **18.** $\frac{\pi}{4}, \frac{3\pi}{4}$. **21.** $\frac{1}{64}$.

HINTS AND SOLUTIONS

4. (i) $\sin^2 54^\circ = \sin^2(90^\circ - 36^\circ) = \cos^2 36^\circ$ $\quad[\because \sin(90^\circ - \theta) = \cos\theta]$

$\sin^2 72^\circ = \sin^2(90^\circ - 18^\circ) = \cos^2 18^\circ$

$$\begin{aligned} \text{L.H.S.} &= \sin^2 54^\circ - \sin^2 72^\circ = \cos^2 36^\circ - \cos^2 18^\circ \\ &= (1 - \sin^2 36^\circ) - (1 - \sin^2 18^\circ) \quad [\because \cos^2\theta = 1 - \sin^2\theta] \\ &= \sin^2 18^\circ - \sin^2 36^\circ = \text{R.H.S.} \end{aligned}$$

(ii) $$\cos^3\frac{5\pi}{8} = \left[\cos\left(\pi - \frac{3\pi}{8}\right)\right]^3 = \left[-\cos\frac{3\pi}{8}\right]^3 = -\cos^3\frac{3\pi}{8}$$

$$\cos^3\frac{7\pi}{8} = \left[\cos\left(\pi - \frac{\pi}{8}\right)\right]^3 = \left[-\cos\frac{\pi}{8}\right]^3 = -\cos^3\frac{\pi}{8}.$$

7. In any quadrilateral $ABCD$, $A + B + C + (D) = 360°$

$\Rightarrow \quad A + B = 360° - (C + D)$

$\Rightarrow \quad \sin(A + B) = \sin[360° - (C + D)] = -\sin(C + D) \quad [\sin(2\pi - \theta) = -\sin\theta]$

$\Rightarrow \quad \sin(A + B) + \sin(C + D) = 0$

Also, $\cos(A + B) = \cos[360° - (C + D)] = \cos(C + D)$ $[\cos(2\pi - \theta) = \cos\theta]$.

9. $\text{cosec}\left(\frac{5\pi}{6}\right) = \text{cosec}\left(\pi - \frac{\pi}{6}\right) = \text{cosec}\ \frac{\pi}{6} = 2$

$\text{cosec}\left(\frac{7\pi}{6}\right) = \text{cosec}\left(\pi + \frac{\pi}{6}\right) = -\text{cosec}\ \frac{\pi}{6} = -2$

$\sin\left(\frac{3\pi}{4}\right) = \sin\left(\pi - \frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

11. In any ΔABC, $A + B + C = 180°$, or $A + B = 180° - C$

(i) $\sin(A + B) = \sin(180° - C) = \sin C$ $\quad [\sin(180° - \theta) = \sin\theta]$

(ii) $\cos(A + B) = \cos(180° - C) = -\cos C$ $\quad [\cos(180° - \theta) = -\cos\theta]$

$\Rightarrow \cos(A + B) + \cos C = 0$

(iii) $\cos\left[\frac{A+B}{2}\right] = \cos\left[\frac{180° - C}{2}\right] = \cos\left[90° - \frac{C}{2}\right] = \sin\frac{C}{2}$

$\therefore \quad \cos\left[\frac{A+B}{2}\right] = \sin\frac{C}{2}.$

(iv) $\tan\frac{A+B}{2} = \tan\left[\frac{180° - C}{2}\right] = \tan\left[90° - \frac{C}{2}\right] = \cot\frac{C}{2}$

$\therefore \quad \tan\frac{A+B}{2} = \cot\frac{C}{2}.$

12. (i) Since A, B, C, D are the angles of a cyclic quadrilateral, we have

$A + C = 180°$ and $B + D = 180°$

$\Rightarrow \quad \cos A = \cos(180° - C)$ and $\cos B = (180° - D)$

$\Rightarrow \quad \cos A = -\cos C$ and $\cos B = -\cos D$ $\quad [\cos(\pi - \theta) = -\cos\theta]$

$\Rightarrow \quad \cos A + \cos B = -(\cos C + \cos D) \Rightarrow \cos A + \cos B + \cos C + \cos D = 0.$

(ii) $\quad$ L.H.S $= -\cos A - \cos B - \cos C - \cos D$

$= -\cos A - \cos B - \cos(180° - A) - \cos(180° - B)$

$= -\cos A - \cos B - (-\cos A) - (-\cos B) = 0.$

13. When n is a positive integer

$\sin[n\pi + (-1)^n x] = \sin[\pi + \{(n - 1)\pi + (-1)^n x\}]$ $\quad [\because \sin(\pi + \theta) = -\sin\theta]$

$= (-1)\sin[(n - 1)\pi + (-1)^n x]$

$= (-1)\sin[\pi + \{(n - 2)\pi + (-1)^n x\}]$

$= (-1)^2\sin[(n - 2)\pi + (-1)^n x] = (-1)^3\sin[(n - 3)\pi + (-1)^n x]$

$= (-1)^n\sin[(n - n)\pi + (-1)^n x] = (-1)^n\sin[(-1)^n x]$

$= (-1)^n(-1)^n\sin x$ $\quad [\because \sin(-1)^n x = (-1)^n\sin x]$

$= (-1)^{2n}\sin x = \sin x$ $\quad [\because (-1)^{2n} = 1]$

14. $\sin(n\pi+\theta) = \sin[\pi+\{(n-1)\pi+\theta\}] = -\sin[(n-1)\pi+\theta]$ $[\because \sin(180°+\theta) = \sin\theta]$

$= -\sin[\pi+\{(n-2)\pi+\theta\}] = (-1)^2\sin[(n-2)\pi+\theta]$

$[\because \sin(180°+\theta) = -\sin\theta]$

$= (-1)^3\sin[(n-3)\pi+\theta] = (-1)^n\sin[(n-n)\pi+\theta] = (-1)^n\sin\theta.$

15. If n is an even integer, let $n = 2k$, where $k \in I$.

Then $\sin\left\{n\pi+(-1)^n.\frac{\pi}{6}\right\} = \sin\left\{2k\pi+(-1)^{2k}.\frac{\pi}{6}\right\} = \sin\left\{2k\pi+\frac{\pi}{6}\right\} = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$.

If n is an odd integer, let $n = 2k+1$, where $k \in I$.

Then $\sin\left\{n\pi+(-1)^n.\frac{\pi}{6}\right\} = \sin\left\{(2k+1)\pi+(-1)^{2k+1}\frac{\pi}{6}\right\}$

$= \sin\left\{2k\pi+\pi-\frac{\pi}{6}\right\} = \sin\left\{\pi-\frac{\pi}{6}\right\} = \sin\frac{\pi}{6} = \frac{1}{2}.$

Thus, $\sin\left\{n\pi+(-1)^n\frac{\pi}{6}\right\} = \frac{1}{2}$ for all integral values of n.

16. $\tan 145° = \tan(180°-35°) = -\tan 35° = -x$

$\tan 125° = \tan(90°+35°) = -\cot 35° = -\frac{1}{\tan 35°} = -\frac{1}{x}.$

17. (i) In first quadrant $\theta_1 > \theta_2 \Rightarrow \sin\theta_1 > \sin\theta_2$, as value of $\sin\theta$ steadily increases from 0 to 1, θ increases from 0° to 90°.

Now, $\cos 40° = \cos(90°-50°) = \sin 50°$ $\therefore$ $\sin 50° > \sin 40° \Rightarrow \cos 40° > \sin 40°$

(ii) $\sin(-400°) = -\sin(400°) = -\sin(360°+40°) = -\sin(40°)$

$\cos(-400°) = \cos(400°) = \cos(360°+40°) = \cos 40° = \cos(90°-50°) = \sin 50°$

$\therefore$ $\sin(-400°)+\cos(-400°) = -\sin 40° + \sin 50° = \sin 50° - \sin 40° > 0.$

$(\because \sin 50° > \sin 40°)$

20. L.H.S. $= \sin(120°-B) = \sin[120°-(60°-C)] = \sin(60°+C)$

$= \sin(180°-(60°+C))$ $[\sin(\pi-\theta) = \sin\theta]$

$= \sin(120°-C) =$ R.H.S.

21. Let $x = \prod_{k=0}^{6}\sin\frac{(2k+1)\pi}{14} = \sin\frac{\pi}{14}.\sin\frac{3\pi}{14}...\sin\frac{7\pi}{14}...\sin\frac{13\pi}{14}$

$= \left(\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\right)^2$ $\left[\because \sin\frac{13\pi}{14} = \sin\frac{\pi}{14}, \sin\frac{\pi}{2} = 1\right]$

But $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14} = \cos\left(\frac{\pi}{2}-\frac{\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{3\pi}{14}\right)\cos\left(\frac{\pi}{2}-\frac{5\pi}{14}\right)$

$= \cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7} = \cos\frac{\pi}{7}\cos\frac{2\pi}{7}\left(-\cos\frac{4\pi}{7}\right)$

$= -\frac{1}{8}\frac{\sin\frac{8\pi}{7}}{\sin\frac{\pi}{7}} = -\frac{1}{8}\frac{\sin\left(\pi+\frac{\pi}{7}\right)}{\sin\frac{\pi}{7}} = \frac{1}{8}$ $\therefore$ $x = \frac{1}{64}.$

22. The given expression

$= \cos^2\frac{\pi}{16}+\cos^2\frac{3\pi}{16}+\cos^2\left(\frac{\pi}{2}-\frac{3\pi}{16}\right)+\cos^2\left(\frac{\pi}{2}-\frac{\pi}{16}\right)$

$= \cos^2\frac{\pi}{16}+\cos^2\frac{3\pi}{16}+\sin^2\frac{3\pi}{16}+\sin^2\frac{\pi}{16}$

$= \left(\cos^2\frac{\pi}{16}+\sin^2\frac{\pi}{16}\right)+\left(\cos^2\frac{3\pi}{16}+\sin^2\frac{3\pi}{16}\right) = 1+1 = 2.$

TRIGONOMETRICAL RATIOS OF SUM AND DIFFERENCE OF ANGLES

Addition Formulae of sin (A + B), cos (A + B), tan (A + B)

To prove geometrically:

(i) $\sin(A + B) = \sin A \cos B + \cos A \sin B$;

(ii) $\cos(A + B) = \cos A \cos B - \sin A \sin B$;

(iii) $\tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$.

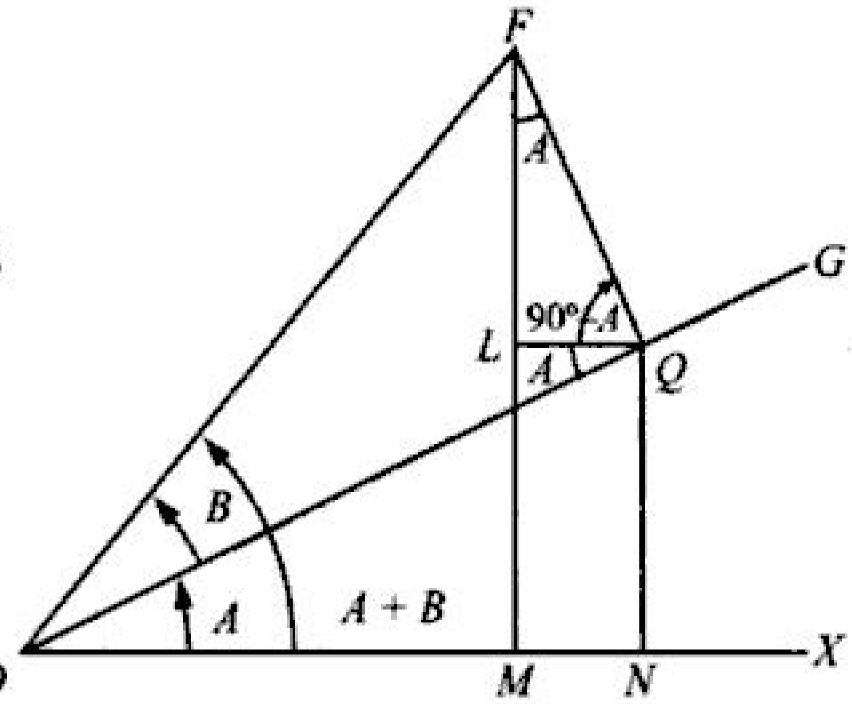

Fig. 3.30

Proof. (i) Refer to Fig. 3.30. Let the revolving line start from initial line OX and trace out $\angle XOG = A$ in the anti-clockwise direction and let the revolving line revolve further to trace out the $\angle GOP = B$, so that $\angle XOP = A + B$. From the point, draw $PM \perp OX$ and $PQ \perp OG$. From Q, draw $QL \perp PM$ and $QN \perp OX$, then

$$\angle QPL = 180° - 90° - \angle PQL$$
$$= 180° - 90° - (90° - A) = A$$

(i) $$\sin(A + B) = \sin \angle XOP = \frac{PM}{OP} = \frac{PL + LM}{OP}$$

$$= \frac{PL + QN}{OP} = \frac{PL}{OP} + \frac{QN}{OP} = \frac{PL}{PQ}\cdot\frac{PQ}{OP} + \frac{QN}{OQ}\cdot\frac{OQ}{OP}$$

$$= \cos A \sin B + \sin A \cos B = \sin A \cos B + \cos A \sin B$$

(ii) $$\cos(A + B) = \frac{OM}{OP} = \frac{ON - MN}{OP} = \frac{ON}{OP} - \frac{MN}{OP} = \frac{ON}{OP} - \frac{LQ}{OP}$$

$$= \frac{ON}{OQ}\cdot\frac{OQ}{OP} - \frac{LQ}{PQ}\cdot\frac{PQ}{OP} = \cos A \cos B - \sin A \sin B$$

(iii) $$\tan(A + B) = \frac{PM}{OM}$$

$$\tan(A + B) = \frac{PM}{OM} = \frac{LM + PL}{OM} = \frac{QN + PL}{ON - MN} = \frac{QN + PL}{ON - LQ}$$

$$= \frac{\dfrac{QN}{ON} + \dfrac{PL}{ON}}{1 - \dfrac{LQ}{ON}} = \frac{\dfrac{QN}{ON} + \dfrac{PL}{ON}}{1 - \dfrac{LQ}{PL}\cdot\dfrac{PL}{ON}} = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\left(\because \frac{PL}{ON} = \frac{PL}{PQ}\cdot\frac{PQ}{OQ}\cdot\frac{OQ}{ON} = \cos A \tan B \sec A = \tan B\right)$$

Note: Expression for $\tan(A + B)$ could be obtained from those of $\sin(A + B)$ and $\cos(A + B)$ as follows:

$$\tan(A + B) = \frac{\sin(A + B)}{\cos(A + B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}$$

Dividing numerator and denominator by cos A cos B, we get

$$\tan(A+B) = \frac{\dfrac{\sin A\cos B}{\cos A\cos B} + \dfrac{\cos A\sin B}{\cos A\cos B}}{\dfrac{\cos A\cos B}{\cos A\cos B} - \dfrac{\sin A\sin B}{\cos A\cos B}} = \frac{\dfrac{\sin A}{\cos A} + \dfrac{\sin B}{\cos B}}{1 - \dfrac{\sin A}{\cos A}\cdot\dfrac{\sin B}{\cos B}} = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

CAUTION

- The formula for tan $(A + B)$ is valid only when none of A, B and $A + B$ is a multiple of $\pi/2$.
- It should be noted that the prefixes sin, cos, tan, etc. are not multipliers; thus

  $\sin(A + B) \neq \sin A + \sin B$;

  $\cos(A + B) \neq \cos A + \cos B$;

  $\tan(A + B) \neq \tan A + \tan B$.

Difference Formulae of sin (*A* – *B*), cos (*A* – *B*), tan (*A* – *B*)

Prove geometrically:

(i) $\sin(A - B) = \sin A\cos B - \cos A\sin B$;

(ii) $\cos(A - B) = \cos A\cos B + \sin A\sin B$;

(iii) $\tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A\tan B}$.

Proof. Let the revolving line starting from its initial position OX trace $\angle XOL = A$ in anti-clockwise direction and revolving back through angle B and occupy the final position OP, so that $\angle XOP = A - B$. From the point P draw $PQ \perp OX$ and $PR \perp OL$.

From R draw $RS \perp OX$ and RT perpendicular on QP(produced).

Now $\angle RPT = 90° - \angle PRT + \angle TRL = \angle A$.

$\therefore$ (i) $\sin(A - B) = \sin\angle XOP = \dfrac{PQ}{OP} = \dfrac{TQ - TP}{OP} = \dfrac{RS - TP}{OP} = \dfrac{RS}{OP} - \dfrac{TP}{OP}$

$$= \frac{RS}{OR}\cdot\frac{OR}{OP} - \frac{TP}{RP}\cdot\frac{RP}{OP} = \sin A\cos B - \cos A\sin B$$

(ii) $\cos(A - B) = \dfrac{OQ}{OP} = \dfrac{OS + SQ}{OP} = \dfrac{OS + RT}{OP}$

$$= \frac{OS}{OP} + \frac{RT}{OP} = \frac{OS}{OR}\cdot\frac{OR}{OP} + \frac{RT}{RP}\cdot\frac{RP}{OP}$$

$= \cos A\cos B + \sin A\sin B$.

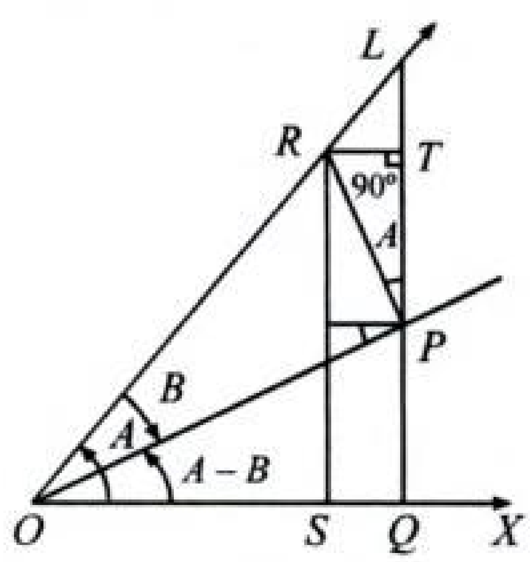

Fig. 3.31

(iii) $\tan(A - B) = \dfrac{PQ}{OQ} = \dfrac{TQ - TP}{OQ}$

$$= \frac{TQ - TP}{OS + SQ} = \frac{RS - TP}{OS + RT}$$

$$= \frac{\dfrac{RS}{OS} - \dfrac{TP}{OS}}{1 + \dfrac{RT}{OS}} = \frac{\dfrac{RS}{OS} - \dfrac{TP}{OS}}{1 + \dfrac{RT}{TP}\cdot\dfrac{TP}{OS}} = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

$$\left(\because \frac{TP}{OS} = \frac{TP}{PR} \cdot \frac{PR}{OR} \cdot \frac{OR}{OS} = \cos A \tan B \sec A = \tan B\right)$$

Note: Expression for $\tan(A - B)$ could be obtained from those of $\sin(A - B)$ and $\cos(A - B)$ as follows:

$$\tan(A - B) = \frac{\sin(A-B)}{\cos(A-B)} = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B}$$

Dividing numerator and denominator by $\cos A \cos B$, we get

$$\tan(A - B) = \frac{\dfrac{\sin A \cos B}{\cos A \cos B} - \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} + \dfrac{\sin A \sin B}{\cos A \cos B}} = \frac{\dfrac{\sin A}{\cos A} - \dfrac{\sin B}{\cos B}}{1 + \dfrac{\sin A \sin B}{\cos A \cos B}} = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

CAUTION

- The formula for $\tan(A - B)$ is valid only when none of A, B and $A - B$ is a multiple of $\pi/2$.
- $\sin(A - B) \neq \sin A - \sin B$;
  $\cos(A - B) \neq \cos A - \cos B$;
  $\tan(A - B) \neq \tan A - \tan B$.

This is because sin, cos and tan are not multipliers.

Cor. 1. (i) $\tan(45° + A) = \dfrac{1 + \tan A}{1 - \tan A}$; (ii) $\tan(45° - A) = \dfrac{1 - \tan A}{1 + \tan A}$.

Proof. (i) $\tan(45° + A) = \dfrac{\tan 45° + \tan A}{1 - \tan 45° \tan A}$ $\left[\because \tan(A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}\right]$

$= \dfrac{1 + \tan A}{1 - 1 . \tan A}$ $[\because \tan 45° = 1]$

(ii) $\tan(45° - A) = \dfrac{\tan 45° - \tan A}{1 + \tan 45° \tan A}$ $\left[\because \tan(A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}\right]$

$= \dfrac{1 - \tan A}{1 + 1 \tan A}$ $[\because \tan 45° = 1]$

Cor. 2. (i) $\cot(A + B) = \dfrac{\cot A \cot B - 1}{\cot A + \cot B}$. (ii) $\cot(A - B) = \dfrac{\cot A \cot B + 1}{\cot B - \cot A}$.

Proof. (i) $\cot(A + B) = \dfrac{\cos(A+B)}{\sin(A+B)} = \dfrac{\cos A \cos B - \sin A \sin B}{\sin A \cos B + \cos A \sin B}$

$$= \frac{\dfrac{\cos A \cos B}{\sin A \sin B} - \dfrac{\sin A \sin B}{\sin A \sin B}}{\dfrac{\sin A \cos B}{\sin A \sin B} + \dfrac{\cos A \sin B}{\sin A \sin B}} = \frac{\dfrac{\cos A}{\sin A} \cdot \dfrac{\cos B}{\sin B} - 1}{\dfrac{\cos B}{\sin B} + \dfrac{\cos A}{\sin A}}$$

[Dividing the numerator and denominator by $\sin A \sin B$]

$$= \frac{\cot A \cot B - 1}{\cot B + \cot A}$$

(ii) $$\cot(A-B) = \frac{\cos(A-B)}{\sin(A-B)} = \frac{\cos A\cos B + \sin A\sin B}{\sin A\cos B - \cos A\sin B}$$

$$= \frac{\dfrac{\cos A\cos B}{\sin A\sin B} + \dfrac{\sin A\sin B}{\sin A\sin B}}{\dfrac{\sin A\cos B}{\sin A\sin B} - \dfrac{\cos A\sin B}{\sin A\sin B}}$$

[Dividing the numerator and denominator by $\sin A \sin B$]

$$= \frac{\dfrac{\cos A}{\sin A}\cdot\dfrac{\cos B}{\sin B} + 1}{\dfrac{\cos B}{\sin B} - \dfrac{\cos A}{\sin A}} = \frac{\cot A\cot B + 1}{\cot B - \cot A}$$

Some Useful Results

(i) $\sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B = \cos^2 B - \cos^2 A$

(ii) $\cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B = \cos^2 B - \sin^2 A$

(iii) $\sin(A+B+C) = \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C$

(iv) $\cos(A+B+C) = \cos A\cos B\cos C - \cos A\sin B\sin C - \sin A\cos B\sin C - \sin A\sin B\cos C$

(v) $$\tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A\tan B\tan C}{1 - \tan A\tan B - \tan B\tan C - \tan C\tan A}$$

Proof. (i) $\sin(A+B)\sin(A-B)$

$= (\sin A\cos B + \cos A\sin B)(\sin A\cos B - \cos A\sin B)$

$= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B = \sin^2 A(1-\sin^2 B) - (1-\sin^2 A)\sin^2 B$

$= \sin^2 A - \sin^2 A\sin^2 B - \sin^2 B + \sin^2 A\sin^2 B = \sin^2 A - \sin^2 B$

$= (1-\cos^2 A) - (1-\cos^2 B) = \cos^2 B - \cos^2 A$

(ii) $\cos(A+B)\cos(A-B)$

$= (\cos A\cos B - \sin A\sin B)(\cos A\cos B + \sin A\sin B)$

$= \cos^2 A\cos^2 B - \sin^2 A\sin^2 B = \cos^2 A(1-\sin^2 B) - (1-\cos^2 A)\sin^2 B$

$= \cos^2 A - \sin^2 B = (1-\sin^2 A) - (1-\cos^2 B) = \cos^2 B - \sin^2 A$

(iii) $\sin(A+B+C) = \sin[(A+B)+C]$

$= \sin(A+B)\cos C + \cos(A+B)\sin C$

$= (\sin A\cos B + \cos A\sin B)\cos C + (\cos A\cos B - \sin A\sin B)\sin C$

$= \sin A\cos B\cos C + \cos A\sin B\cos C + \cos A\cos B\sin C - \sin A\sin B\sin C$

(iv) $\cos(A+B+C) = \cos[(A+B)+C]$

$= \cos(A+B)\cos C - \sin(A+B)\sin C$

$= (\cos A\cos B - \sin A\sin B)\cos C - (\sin A\cos B + \cos A\sin B)\sin C$

$= \cos A\cos B\cos C - \sin A\sin B\cos C - \sin A\cos B\sin C - \cos A\sin B\sin C$

(v) $\tan(A+B+C) = \tan((A+B)+C)$

$$= \frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C} = \frac{\dfrac{\tan A+\tan B}{1-\tan A\tan B}+\tan C}{1-\left(\dfrac{\tan A+\tan B}{1-\tan A\tan B)}\right)\tan C}$$

$$= \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$$

Example 1. *Prove that*

(i) $\sin 75° = \dfrac{\sqrt{6}+\sqrt{2}}{4}$; (ii) $\cos 75° = \dfrac{\sqrt{6}-\sqrt{2}}{4}$;

(iii) $\tan 75° + \cot 75° = 4$; (iv) $\tan 15° = 2-\sqrt{3}$

Solution: (i) $\sin 75° = \sin(45°+30°) = \sin 45° \cos 30° + \cos 45° \sin 30°$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\cdot\frac{1}{2}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$

[Multiplying Num. and Den. by $\sqrt{2}$]

$$= \frac{\sqrt{6}+\sqrt{2}}{4}$$

(ii) $\cos 75° = \cos(45°+30) = \cos 45° \cos 30° - \sin 45° \sin 30°$

$$= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\cdot\frac{1}{2}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$

[Multiplying Num. and Den. by $\sqrt{2}$]

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

(iii) $\tan 75° = \tan(45°+30°) = \dfrac{\tan 45°+\tan 30°}{1-\tan 45°\tan 30°} = \dfrac{1+\dfrac{1}{\sqrt{3}}}{1-\dfrac{1}{\sqrt{3}}} = \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$ ...(1)

and $\cot 75° = \dfrac{1}{\tan 75°} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$ ...(2)

$$\therefore \tan 75° + \cot 75° = \frac{\sqrt{3}+1}{\sqrt{3}-1}+\frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}+1)^2+(\sqrt{3}-1)^2}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$= \frac{3+1+2\sqrt{3}+3+1-2\sqrt{3}}{3-1} = \frac{8}{2} = 4$$

(iv) $\tan 15° = \tan(45° - 30°) = \dfrac{\tan 45° - \tan 30°}{1 + \tan 45° \tan 30°}$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \times \frac{1}{\sqrt{3}}} = \frac{(\sqrt{3}-1)/\sqrt{3}}{(\sqrt{3}+1)/\sqrt{3}}$$

$$= \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1} = \frac{(\sqrt{3}-1)^2}{3-1}$$

$$= \frac{4-2\sqrt{3}}{2} = 2 - \sqrt{3}$$

Example 2. *If* $\sin \alpha = \dfrac{15}{17}$ *and* $\cos \beta = \dfrac{12}{13}$, *find the values of* $\sin(\alpha + \beta)$ α, *and* β, *are positive acute angles.*

Solution: Given $\sin \alpha = \dfrac{15}{17}$

$\therefore$ $\cos^2\alpha = 1 - \sin^2\alpha = 1 - \left(\dfrac{15}{17}\right)^2 = 1 - \dfrac{225}{289} = \dfrac{289 - 225}{289} = \dfrac{64}{289}$

$\therefore$ $\cos \alpha = \dfrac{8}{17}$ [$\because$ α is an acute angle $\Rightarrow \cos \alpha > 0$]

Now $\cos \beta = \dfrac{12}{13}$

$\therefore$ $\sin^2\beta = 1 - \cos^2\beta = 1 - \left(\dfrac{12}{13}\right)^2 = 1 - \dfrac{144}{169} = 169 - \dfrac{144}{169} = \dfrac{25}{169}$

$\therefore$ $\sin \beta = \dfrac{5}{13}$ [$\because$ β is an acute angle $\Rightarrow \sin \beta > 0$]

Now, $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$

$$= \frac{15}{17} \times \frac{12}{13} + \frac{8}{17} \times \frac{5}{13} = \frac{180 + 40}{17 \times 13} = \frac{220}{221}.$$

Example 3. *Prove that*

$$\frac{\sin(B-C)}{\cos B \cos C} + \frac{\sin(C-A)}{\cos C \cos A} + \frac{\sin(A-B)}{\cos A \cos B} = 0$$

Solution: First term of L.H.S. $= \dfrac{\sin(B-C)}{\cos B \cos C} = \dfrac{\sin B \cos C - \cos B \sin C}{\cos B \cos C}$

$$= \frac{\sin B \cos C}{\cos B \cos C} - \frac{\cos B \sin C}{\cos B \cos C} = \tan B - \tan C$$

Similarly, Second term of L.H.S. $= \tan C - \tan A$

and, Third term of L.H.S. $= \tan A - \tan B$

Now, L.H.S. $= (\tan B - \tan C) + (\tan C - \tan A) + (\tan A - \tan B) = 0.$

Example 4. *Find the value of each of the following:*

(i) $\sin 63^\circ \cos 27^\circ + \cos 63^\circ \sin 27^\circ$; (ii) $\cos 72^\circ \cos 42^\circ + \sin 72^\circ \sin 42^\circ$;

(iii) $\dfrac{\tan 47^\circ + \tan 43^\circ}{1 - \tan 47^\circ \tan 43^\circ}$

Solution: (i) We know that

$$\sin A \cos B + \cos A \sin B = \sin (A + B)$$

$$\therefore \quad \sin 63^\circ \cos 27^\circ + \cos 63^\circ \sin 27 = \sin (63^\circ + 27^\circ) = \sin 90^\circ = 1$$

(ii) $\cos A \cos B + \sin A \sin B = \cos (A - B)$

$$\therefore \quad \cos 72^\circ \cos 42^\circ + \sin 72^\circ \sin 42^\circ = \cos (72^\circ - 42^\circ) = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

(iii) $\because \dfrac{\tan A + \tan B}{1 - \tan A \tan B} = \tan (A + B)$

$$\therefore \quad \frac{\tan 47^\circ + \tan 43^\circ}{1 - \tan 47^\circ \tan 43^\circ} = \tan (47^\circ + 43^\circ) = \tan 90^\circ = \infty.$$

Example 5. *If* $\cos A = \dfrac{1}{7}$; $\cos B = \dfrac{13}{14}$; *prove that* $A - B = 60^\circ$; A *and* B *being +ve acute angles.*

Solution: Since $\cos A = \dfrac{1}{7}$

$$\therefore \quad \sin^2 A = 1 - \cos^2 A = 1 - \frac{1}{49} = \frac{48}{49} \quad \therefore \quad \sin A = \frac{4\sqrt{3}}{7} \qquad [\because A \text{ is acute}]$$

and $\cos B = \dfrac{13}{14}$ [Given]

$$\therefore \quad \sin B = \sqrt{1 - \cos^2 B} = \sqrt{1 - \frac{169}{196}} = \sqrt{\frac{27}{196}} = \frac{3\sqrt{3}}{14}$$

$$\therefore \quad \cos (A - B) = \cos A \cos B + \sin A \sin B$$

$$= \frac{1}{7} \cdot \frac{13}{14} + \frac{4\sqrt{3}}{7} \cdot \frac{3\sqrt{3}}{14} = \frac{13 + 36}{98} = \frac{49}{98} = \frac{1}{2} = \cos 60^\circ$$

$$\therefore \quad A - B = 60^\circ$$

Example 6. *Prove that*

(i) $\sin (45^\circ + A) \cos (45^\circ - B) + \cos (45^\circ + A) \sin (45^\circ - B) = \cos (A - B)$

(ii) $\sin (n + 1) A \sin (n - 1)A + \cos (n + 1)A \cos (n - 1) A = \cos 2A$.

Solution:

(i) L.H.S. $= \sin (45^\circ + A) \cos (45^\circ - B) + \cos (45^\circ + A) \sin (45^\circ - B)$

$= \sin [(45^\circ + A) + (45^\circ - B)]$ $[\because \sin A \cos B + \cos A \sin B = \sin (A + B)]$

$= \sin [90^\circ + (A - B)] = \cos (A - B) =$ R.H.S. $[\because \sin (90^\circ + \theta) = \cos \theta]$

(ii) L.H.S. $= \sin (n + 1)A \sin (n - 1)A + \cos (n + 1)A \cos (n - 1)A$

$= \cos (n + 1)A \cos (n - 1)A + \sin (n + 1)A \sin (n - 1)A$

$= \cos [(n + 1)A - (n - 1)A]$ $[\because \cos A \cos B + \sin A \sin B = \cos (A - B)]$

$= \cos (nA + A - nA + A) = \cos 2A =$ R.H.S.

Example 7. *Show that* sin 105° + cos 105° = cos 45°.

Solution: $\sin 105° + \cos 105° = \sin(60° + 45°) + \cos(60° + 45°)$

$$= (\sin 60° \cos 45° + \cos 60° \sin 45°) + (\cos 60° \cos 45° - \sin 60° \sin 45°)$$

$$= \frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}+\frac{1}{2}\times\frac{1}{\sqrt{2}}+\frac{1}{2}\times\frac{1}{\sqrt{2}}-\frac{\sqrt{3}}{2}\times\frac{1}{\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}}+\frac{1}{2\sqrt{2}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}} = \cos 45°.$$

Example 8. *Prove that*

(i) $\dfrac{\cos 17° + \sin 17°}{\cos 17° - \sin 17°} = \tan 62°$; (ii) $\tan 50° = \tan 40° + 2\tan 10°$.

Solution: (i) Dividing numerator and denominator of L.H.S. by cos 17°, we get

$$\text{L.H.S.} = \frac{\dfrac{\cos 17°}{\cos 17°}+\dfrac{\sin 17°}{\cos 17°}}{\dfrac{\cos 17°}{\cos 17°}-\dfrac{\sin 17°}{\cos 17°}} = \frac{1+\tan 17°}{1-\tan 17°} = \frac{\tan 45° + \tan 17°}{1-\tan 45°.\tan 17°}$$

$$= \tan(45° + 17°) = \tan 62° = \text{R.H.S.}$$

(ii) $\tan 50° = \tan(40° + 10°)$ or $\tan 50° = \dfrac{\tan 40° + \tan 10°}{1-\tan 40°\cdot\tan 10°}$

Cross-multiplying,

$$\tan 50° - \tan 50°\cdot\tan 40°\cdot\tan 10° = \tan 40° + \tan 10°$$

$\therefore$ $\tan 50° = \tan 40° + \tan 10° + \tan 50° \tan 40° \tan 10°$

$$= \tan 40° + \tan 10° + \cot 40° \tan 40° \tan 10°$$

$$= \tan 40° + \tan 10° + \tan 10° = \tan 40° + 2\tan 10° = \text{R.H.S}$$

Example 9. *Prove that*

$$\sin(2n+1)A\cdot\sin A = \sin^2(n+1)A - \sin^2 nA$$

Solution: R.H.S. $= \sin^2(n+1)A - \sin^2 nA = \sin\{(n+1)A + nA\}\cdot\sin\{(n+1)A - nA\}$

$[\because \sin^2 A - \sin^2 B = \sin(A+B)\times\sin(A-B)]$

$$= \sin(2n+1)A.\ \sin A = \text{L.H.S.}$$

Example 10. *If* $\sin\theta = 1/\sqrt{10}$, $\sin\phi = 1/\sqrt{5}$, θ and ϕ *being positive and acute, show that* $\theta + \phi = \pi/4$.

Solution: As θ and ϕ are positive and acute, therefore, all their *t*-ratios are positive.

Then $\cos^2\theta = 1 - \sin^2\theta = 1 - \dfrac{1}{10} = \dfrac{9}{10}$ $\therefore$ $\cos\theta = \dfrac{3}{\sqrt{10}}$

$\cos^2\phi = 1 - \sin^2\phi = 1 - \dfrac{1}{5} = \dfrac{4}{5}$ $\therefore$ $\cos\phi = \dfrac{2}{\sqrt{5}}$

Now, $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

$$= \frac{1}{\sqrt{10}}\cdot\frac{2}{\sqrt{5}}+\frac{3}{\sqrt{10}}\cdot\frac{1}{\sqrt{5}} = \frac{2+3}{\sqrt{50}} = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$\therefore$ $\sin(\theta+\phi) = \sin\dfrac{\pi}{4} \Rightarrow \theta + \phi = \dfrac{\pi}{4}$

Example 11. *If* $\sin\alpha = \frac{15}{17}$ and $\cos\beta = \frac{12}{13}$, *$\alpha$ and β being positive and acute, find the values of* $\sin(\alpha+\beta)$, $\cos(\alpha-\beta)$ and $\tan(\alpha+\beta)$.

Solution: $\sin\alpha = \frac{15}{17}$

$\therefore$ $\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\left(\frac{15}{17}\right)^2} = \sqrt{1-\frac{225}{289}} = \sqrt{\frac{64}{289}} = \frac{8}{17}$

Again $\cos\beta = \frac{12}{13}$

$\therefore$ $\sin\beta = \sqrt{1-\cos^2\beta}\ \sqrt{1-\left(\frac{12}{13}\right)^2} = \sqrt{1-\frac{144}{169}} = \sqrt{\frac{25}{169}} = \frac{5}{13}$

$\therefore$ $\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{5/17}{8/17} = \frac{15}{8}$ and $\tan\beta = \frac{\sin\beta}{\cos\beta} = \frac{5/13}{12/13} = \frac{5}{12}$

Now, $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$$= \frac{15}{17}\times\frac{12}{13}+\frac{8}{17}\times\frac{5}{13} = \frac{180+40}{221} = \frac{220}{221}.$$

Again $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$= \frac{8}{17}\times\frac{12}{13}+\frac{15}{17}\times\frac{5}{13} = \frac{96+75}{221} = \frac{171}{221}$$

and $$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} = \frac{\frac{15}{8}+\frac{5}{12}}{1-\frac{15}{8}\times\frac{5}{12}} = \frac{(180+40)/96}{(96-75)/96} = \frac{220}{21}$$

Hence $\sin(\alpha+\beta) = \frac{220}{221}$, $\cos(\alpha-\beta) = \frac{171}{221}$ and $\tan(\alpha+\beta) = \frac{220}{21}$

Example 12. *Show that*

(i) $\tan 70° = 2\tan 50° + \tan 20°$; (ii) $\tan 3A - \tan 2A - \tan A = \tan 3A \tan 2A \tan A$.

Solution:

(i) $\tan 70° = \tan(50° + 20°)$ $\therefore$ $\tan 70° = \frac{\tan 50° + \tan 20°}{1-\tan 50°\tan 20°}$

$\Rightarrow$ $\tan 70° - \tan 70°\tan 50°\tan 20° = \tan 50° + \tan 20°$

$\Rightarrow$ $\tan 70° - \frac{1}{\tan 20°}\tan 50°\tan 20° = \tan 50° + \tan 20°$

$$\left[\because \tan 70° = \tan(90° - 20°) = \cot 20° = \frac{1}{\tan 20°}\right]$$

$\Rightarrow$ $\tan 70° - \tan 50° = \tan 50° + \tan 20° \Rightarrow \tan 70° = 2\tan 50° + \tan 20°$

(ii) We have

$$\tan 3A = \tan(2A + A) = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$$

$\Rightarrow$ $\tan 3A[1 - \tan 2A \tan A] = \tan 2A + \tan A$

$\Rightarrow$ $\tan 3A - \tan 3A \tan 2A \tan A = \tan 2A + \tan A$

$\Rightarrow$ $\tan 3A - \tan 2A - \tan A = \tan 3A \tan 2A \tan A.$

Example 13. *Prove that* $\dfrac{\sqrt{3}\cos 23^\circ - \sin 23^\circ}{2} = \cos 53^\circ.$

Solution: L.H.S. $= \dfrac{\sqrt{3}\cos 23^\circ - \sin 23^\circ}{2} = \dfrac{\sqrt{3}}{2}\cos 23^\circ - \dfrac{1}{2}\sin 23^\circ$

$= \cos 30^\circ \cos 23^\circ - \sin 30^\circ \sin 23^\circ$

$$\left[\because \frac{\sqrt{3}}{2} = \cos 30^\circ \text{ and } \frac{1}{2} = \sin 30^\circ\right]$$

$= \cos(30^\circ + 23^\circ) = \cos 53^\circ =$ R.H.S.

Example 14. *Prove that* $\dfrac{\tan(45^\circ + x)}{\tan(45^\circ - x)} = \left(\dfrac{1 + \tan x}{1 - \tan x}\right)^2$

Solution: $\tan(45^\circ + x) = \dfrac{\tan 45^\circ + \tan x}{1 - \tan 45^\circ \tan x} = \dfrac{1 + \tan x}{1 - \tan x}$ $\quad (\because \tan 45^\circ = 1)$

$\tan(45^\circ - x) = \dfrac{\tan 45^\circ + \tan x}{1 - \tan 45^\circ \tan x} = \dfrac{1 - \tan x}{1 + \tan x}$

$\therefore$ L.H.S. $= \dfrac{\tan(45^\circ + x)}{\tan(45^\circ - x)} = \dfrac{\left(\dfrac{1 + \tan x}{1 - \tan x}\right)}{\left(\dfrac{1 - \tan x}{1 + \tan x}\right)} = \left(\dfrac{1 + \tan x}{1 - \tan x}\right)^2 =$ R.H.S.

Example 15. *Prove that* $= \dfrac{\tan\left(\dfrac{\pi}{4} + A\right) + \tan\left(\dfrac{\pi}{4} - A\right)}{\tan\left(\dfrac{\pi}{4} + A\right) - \tan\left(\dfrac{\pi}{4} - A\right)} = \operatorname{cosec} 2A.$

Solution: Let $\dfrac{\pi}{4} + A = \alpha$ and $\dfrac{\pi}{4} - A = \beta$

L.H.S. $= \dfrac{\tan\alpha + \tan\beta}{\tan\alpha - \tan\beta} = \dfrac{\dfrac{\sin\alpha}{\cos\alpha} + \dfrac{\sin\beta}{\cos\beta}}{\dfrac{\sin\alpha}{\cos\alpha} - \dfrac{\sin\beta}{\cos\beta}}$

(Multiplying num. and denom. by $\cos\alpha\cos\beta$)

$= \dfrac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\alpha\cos\beta - \cos\alpha\sin\beta} = \dfrac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \dfrac{\sin\left(\dfrac{\pi}{4} + A + \dfrac{\pi}{4} - A\right)}{\sin\left(\dfrac{\pi}{4} + A - \dfrac{\pi}{4} + A\right)}$

$= \dfrac{\sin\dfrac{\pi}{2}}{\sin 2A} = \dfrac{1}{\sin 2A} = \operatorname{cosec} 2A =$ R.H.S.

Example 16. *Prove that*

(i) $\dfrac{\tan 69^\circ + \tan 66^\circ}{1 - \tan 69^\circ \tan 66^\circ} = -1$; (ii) $\dfrac{\tan(A-B) + \tan B}{1 - \tan(A-B)\tan B} = \tan A$.

Solution:

(i) $\dfrac{\tan 69^\circ + \tan 66^\circ}{1 - \tan 69^\circ \tan 66^\circ} = \tan(69^\circ + 66^\circ) = \tan 135^\circ = \tan(90^\circ + 45^\circ) = \cot 45^\circ = -1$

$$\left[\text{Using } \frac{\tan A + \tan B}{1 - \tan A \tan B} = \tan(A+B)\right]$$

(ii) L.H.S. $= \dfrac{\tan(A-B) + \tan B}{1 - \tan(A-B)\tan B} = \tan[(A-B)+B] = \tan A =$ R.H.S.

Example 17. (i) *If* $A + B = 45^\circ$, *prove that* $(1 + \tan A)(1 + \tan B) = 2$

(ii) *If* $A - B = 45^\circ$, *prove that* $(1 + \tan A)(1 + \tan B) = 2\tan A$.

(iii) *If* $A + B = 45^\circ$, *prove that* $(\cot A - 1)(\cot B - 1) = 2$.

Solution: (i) Since $A + B = 45^\circ$, we get $\tan(A+B) = \tan 45^\circ$.

$\Rightarrow \quad \dfrac{\tan A + \tan B}{1 - \tan A \tan B} = 1 \Rightarrow \tan A + \tan B = 1 - \tan A \tan B$

$\Rightarrow \quad \tan A + \tan B + \tan A \tan B = 1$

$\Rightarrow \quad 1 + \tan A + (1 + \tan A)\tan B = 2$ [Adding 1 to both the sides]

$\Rightarrow \quad (1 + \tan A)(1 + \tan B) = 2$

(ii) Since $A - B = 45^\circ$, we get $\tan(A - B) = \tan 45^\circ \Rightarrow \dfrac{\tan A - \tan B}{1 + \tan A \tan B} = 1$

$\Rightarrow \quad \tan A - \tan B = 1 + \tan A \tan B$ ...(1)

$\Rightarrow \quad 1 + \tan A \tan B + \tan B = \tan A$

$\Rightarrow \quad 1 + \tan A + \tan B(1 + \tan A) = 2\tan A$ [Adding $\tan A$ to both the sides]

$\Rightarrow \quad (1 + \tan A)(1 + \tan B) = 2\tan A$.

(iii) L.H.S. $= (\cot A - 1)(\cot B - 1)$

$$= \left(\frac{1}{\tan A} - 1\right)\left(\frac{1}{\tan B} - 1\right) = \left(\frac{1}{\tan A} - 1\right)\left[\frac{1}{\tan(45^\circ - A)} - 1\right]$$

$$= \left(\frac{1 - \tan A}{\tan A}\right)\left[\frac{1 + \tan 45^\circ \tan A}{\tan 45^\circ - \tan A} - 1\right] = \left(\frac{1 - \tan A}{\tan A}\right)\left[\frac{1 + \tan A}{1 - \tan A} - 1\right]$$

$$= \left(\frac{1 - \tan A}{\tan A}\right)\left[\frac{1 + \tan A - 1 + \tan A}{1 - \tan A}\right] = \left(\frac{1 - \tan A}{\tan A}\right)\left(\frac{2\tan A}{1 - \tan A}\right)$$

$= 2 =$ R.H.S.

Example 18. *Prove that* $\sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right)=\frac{1}{\sqrt{2}}\sin A.$

Solution: L.H.S. $=\sin^2\left(\frac{\pi}{8}+\frac{A}{2}\right)-\sin^2\left(\frac{\pi}{8}-\frac{A}{2}\right)$

$$=\sin\left[\left(\frac{\pi}{8}+\frac{A}{2}\right)+\left(\frac{\pi}{8}-\frac{A}{2}\right)\right]\sin\left[\left(\frac{\pi}{8}+\frac{A}{2}\right)-\left(\frac{\pi}{8}-\frac{A}{2}\right)\right]$$

$[\because \sin^2 A-\sin^2 B=\sin(A+B)\sin(A-B)]$

$$=\sin\frac{\pi}{4}\sin A=\frac{1}{\sqrt{2}}\sin A=\text{R.H.S.}$$

Example 19. *Prove that* $\frac{\tan(A+B)}{\cot(A-B)}=\frac{\sin^2 A-\sin^2 B}{\cos^2 A-\sin^2 B}.$

Solution: R.H.S. $=\frac{\sin^2 A-\sin^2 B}{\cos^2 A-\sin^2 B}=\frac{\sin(A+B)\sin(A-B)}{\cos(A+B)\cos(A-B)}$

$$=\frac{\sin(A+B)}{\cos(A+B)}\cdot\frac{\sin(A-B)}{\cos(A-B)}=\tan(A+B)\cdot\tan(A-B)$$

$$=\tan(A+B)\cdot\frac{1}{\cot(A-B)}=\frac{\tan(A+B)}{\cot(A-B)}=\text{L.H.S.}$$

Example 20. *Prove that* $\cos^2(45°+x)-\sin^2(45°-x)$ *is independent of* x.

Solution: $\cos^2(45°+x)-\sin^2(45°-x)$

$$=\cos[(45°+x)+(45°-x)]\cos[(45°+x)-(45°-x)]$$

$[\because \cos^2 A-\sin^2 B=\cos(A+B)\cos(A-B)]$

$$=\cos 90°\cos 2x=(0)\cos 2x=0$$

which does not contain x and hence is independent of x.

Example 21. *If* $\tan\beta=\frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}$*; prove that* $\tan(\alpha-\beta)=(1-n)\tan\alpha$

Solution: L.H.S. $=\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

Putting $\tan\beta=\frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}$, we have

$$\text{L.H.S.}=\frac{\frac{\sin\alpha}{\cos\alpha}-\frac{m\sin\alpha\cos\alpha}{1-n\sin^2\alpha}}{1+\frac{\sin\alpha}{\cos\alpha}\cdot\frac{n\sin\alpha\cos\alpha}{1-n\sin^2\alpha}}\qquad\left[\because \tan\alpha=\frac{\sin\alpha}{\cos\alpha}\right]$$

$$=\frac{\sin\alpha(1-n\sin^2\alpha)-n\sin\alpha\cos^2\alpha}{\cos\alpha(1-n\sin^2\alpha)+n\sin^2\alpha\cos\alpha}=\frac{\sin\alpha-n\sin^3\alpha-n\sin\alpha\cos^2\alpha}{\cos\alpha-n\sin^2\alpha+n\sin^2\alpha\cos\alpha}$$

$$=\frac{\sin\alpha-n\sin\alpha(\sin^2 a+\cos^2\alpha)}{\cos\alpha}=\frac{\sin\alpha-n\sin\alpha}{\cos\alpha}\qquad[\because \sin^2\alpha+\cos^2\alpha=1]$$

$$=\frac{\sin\alpha(1-n)}{\cos\alpha}=(1-n)\tan\alpha$$

Example 22. *If* $3 \tan\theta \tan\varphi = 1$, *prove that* $2\cos(\theta + \varphi) = \cos(\theta - \varphi)$

Solution: $3 \tan\theta \tan\varphi = 1$ or, $\cot\theta \cdot \cot\varphi = 3$

or $$\frac{\cos\theta\cos\varphi}{\sin\alpha\sin\varphi} = \frac{3}{1}$$

By componendo and dividendo, we have

$$\frac{\cos\theta\cos\varphi + \sin\theta\sin\varphi}{\cos\theta\cos\varphi - \sin\theta\sin\varphi} = \frac{3+1}{3-1} \text{ or } \frac{\cos(\theta-\varphi)}{\cos(\theta+\varphi)} = 2$$

or $$2\cos(\theta + \varphi) = \cos(\theta - \varphi).$$

Example 23. *If* $a \tan\left(\theta - \frac{\pi}{6}\right) = b\tan\left(\theta + \frac{2\pi}{3}\right)$, *prove that* $\cos 2\theta = \frac{a+b}{2(a-b)}$.

Solution: $a\tan\left(\theta - \frac{\pi}{6}\right) = b\tan\left(\theta + \frac{2\pi}{3}\right)$

$$\Rightarrow \frac{a}{b} = \frac{\tan\left(\theta + \frac{2\pi}{3}\right)}{\tan\left(\theta - \frac{\pi}{6}\right)} \Rightarrow \frac{a}{b} = \frac{\sin\left(\theta + \frac{2\pi}{3}\right)\cos\left(\theta - \frac{\pi}{6}\right)}{\cos\left(\theta + \frac{2\pi}{3}\right)\sin\left(\theta - \frac{\pi}{6}\right)}$$

By componendo and dividendo, we have

$$\frac{a+b}{a-b} = \frac{\sin\left(\theta+\frac{2\pi}{3}\right)\cos\left(\theta-\frac{\pi}{6}\right) + \cos\left(\theta+\frac{2\pi}{3}\right)\sin\left(\theta-\frac{\pi}{6}\right)}{\sin\left(\theta+\frac{2\pi}{3}\right)\cos\left(\theta-\frac{\pi}{6}\right) - \cos\left(\theta+\frac{2\pi}{3}\right)\sin\left(\theta-\frac{\pi}{6}\right)}$$

$$= \frac{\sin\left[\left(\theta+\frac{2\pi}{3}\right)+\left(\theta-\frac{\pi}{6}\right)\right]}{\sin\left[\left(\theta+\frac{2\pi}{3}\right)-\left(\theta-\frac{\pi}{6}\right)\right]} = \frac{\sin\left(\frac{\pi}{2}+2\theta\right)}{\sin\frac{5\pi}{6}} = \frac{\cos 2\theta}{\sin\left(\pi - \frac{\pi}{6}\right)}$$

$$= \frac{\cos 2\theta}{\sin\frac{\pi}{6}} = \frac{\cos 2\theta}{\frac{1}{2}} \Rightarrow \frac{a+b}{a-b} = 2\cos 2\theta$$

Hence $$\cos 2\theta = \frac{a+b}{2(a-b)}$$

Example 24. *If* $\theta + \phi = \alpha$ and $\tan\theta = k\tan\phi$, *then prove that* $\sin(\theta - \phi) = \frac{k-1}{k+1}\sin\alpha$

Solution: We have $\tan\theta = k\tan\phi$.

$$\therefore \frac{\tan 6}{\tan\phi} = \frac{k}{1} \Rightarrow \frac{\tan 6 + \tan\phi}{\tan\theta - \tan\phi} = \frac{k+1}{k-1} \quad \text{(By componendo and dividendo)}$$

$$\Rightarrow \quad \frac{\frac{\sin\theta}{\cos\theta}+\frac{\sin\phi}{\cos\phi}}{\frac{\sin\theta}{\cos\theta}-\frac{\sin\phi}{\cos\phi}}=\frac{k+1}{k-1} \Rightarrow \frac{\sin\theta\cos\phi+\cos\theta\sin\phi}{\sin\theta\cos\phi-\cos\theta\sin\phi}=\frac{k+1}{k-1}$$

$$\Rightarrow \quad \frac{\sin(\theta+\phi)}{\sin(\theta-\phi)}=\frac{k+1}{k-1} \Rightarrow \frac{\sin(\theta-\phi)}{\sin\alpha}=\frac{k-1}{k+1} \qquad [\because \alpha=\theta+\phi]$$

$$\Rightarrow \quad \sin(\theta-\phi)=\frac{k-1}{k+1}\sin\alpha.$$

Example 25. *If* $\sin A = \frac{3}{5}$, $0 < A < \frac{\pi}{2}$ *and* $\cos B = \frac{-12}{13}$, *and* $\cos B < \frac{3\pi}{2}$, *find the following*:

(i) $\sin(A-B)$; (ii) $\cos(A+B)$; (iii) $\tan(A-B)$.

Solution: We have $\sin A = \frac{3}{5}$, where $0 < A < \frac{\pi}{2}$.

$\therefore \quad \cos A = \pm\sqrt{1-\sin^2 A} \Rightarrow \cos A = +\sqrt{1-\sin^2 A} = \sqrt{1-\frac{9}{25}} = \frac{4}{5}$

In the first quadrant, tangent function is positive. Therefore,

$$\tan A = \frac{\sin A}{\cos A} = \frac{3}{4}$$

It is given that $\cos B = -\frac{12}{13}$ and $\pi < B < \frac{3\pi}{2}$.

$\therefore \quad \sin B = \pm\sqrt{1-\cos^2 B} \quad \Rightarrow \quad \sin B = -\sqrt{1-\cos^2 B}$

[$\because$ sine is negative in the third quadrant.]

$$\Rightarrow \quad \sin B = -\sqrt{1-\left(\frac{-12}{13}\right)^2} = -\frac{5}{13}$$

In the third quadrant tangent function is positive. Therefore,

$$\tan B = \frac{\sin B}{\cos B} = \frac{5}{12}$$

Now,

(i) $\sin(A-B) = \sin A\cos B - \cos A\sin B = \frac{3}{5}\times\frac{-12}{13}-\frac{4}{5}\times\frac{-5}{13} = \frac{-16}{65}$

(ii) $\cos(A+B) = \cos A\cos B - \sin A\sin B = \frac{4}{5}\times\frac{-12}{13}-\frac{3}{5}\times\frac{-5}{13} = \frac{-33}{65}$

(iii) $\tan(A-B) = \dfrac{\tan A-\tan B}{1+\tan A\tan B} = \dfrac{\frac{3}{4}-\frac{5}{12}}{1+\frac{3}{4}\times\frac{5}{12}} = \dfrac{16}{63}$

Example 26. *Find the value of* $\tan(\alpha+\beta)$, *given that*

$\cot\alpha = \frac{1}{2}$, $\alpha \in \left(\pi, \frac{3\pi}{2}\right)$ *and* $\sec\beta = \frac{5}{3}$, $\beta \in \left(\frac{\pi}{2}, \pi\right)$.

Solution: $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

Given, $\cot\alpha = \frac{1}{2} \Rightarrow \tan\alpha = 2$

Also $\sec\alpha = -\frac{5}{3}$

$$\tan\beta = \sqrt{\sec^2\beta - 1} = \pm\sqrt{\frac{25}{9} - 1} = \pm\frac{4}{3}$$

But $\beta \in \left(\frac{\pi}{2}, \pi\right) \Rightarrow \tan\beta = -\frac{4}{3}$ $\quad\therefore\quad \tan(\alpha+\beta) = \dfrac{2 - \frac{4}{3}}{1 - 2\left(-\frac{4}{3}\right)} = \dfrac{6-4}{3+8} = \dfrac{2}{11}.$

EXERCISE 3.4

LEVEL OF DIFFICULTY A

1. Evaluate:

(i) $\sin 22^\circ \cos 38^\circ + \cos 22^\circ \sin 38^\circ$; (ii) $\cos 40^\circ \cos 20^\circ - \sin 40^\circ \sin 20^\circ$;

(iii) $\sin 40^\circ \cos 10^\circ - \cos 40^\circ \sin 10^\circ$; (iv) $\cos 130^\circ \cos 40^\circ + \sin 130^\circ \sin 40^\circ$;

(v) $\dfrac{\tan A - 1}{\tan A + 1}$ if $A = 75^\circ$; (vi) $\sin\dfrac{7\pi}{12}\cos\dfrac{\pi}{4} - \cos\dfrac{7\pi}{12}\sin\dfrac{\pi}{4}$;

(vii) $\sin\dfrac{\pi}{4}\cos\dfrac{\pi}{12} + \cos\dfrac{\pi}{4}\sin\dfrac{\pi}{12}$; (viii) $\cos\dfrac{2\pi}{3}\cos\dfrac{\pi}{4} - \sin\dfrac{2\pi}{3}\sin\dfrac{\pi}{4}$.

2. Find the value of each of the following:

(i) $\sin A \sin(B+C) - \cos A \cos(B+C)$; (ii) $\sin A \cos B + \cos A \sin B$ if $A + B = \pi$;

(iii) $\sin(70^\circ + A)\cos(20^\circ - A) + \cos(70^\circ + A)\sin(20^\circ - A)$;

(iv) $\tan(A + B)$, if $\tan A = 1/2$, $\tan B = 1/3$;

(v) $\sin(A - B)$, if $\cos A = 12/13$, $\cos B = 8/17$, $0 < A, B < \pi/2$;

(vi) $\sin(A - B)$, $\sin(A + B)$, $\cos(A - B)$, $\cos(A + B)$, if $\sin A = \dfrac{3}{5}$ and $\cos B = \dfrac{9}{41}$, $0 < A, B < \dfrac{\pi}{2}$

(vii) $\cos(A + B)$, $\sin(A - B)$, if $\cos A = \dfrac{4}{5}$, $\cos B = \dfrac{12}{13}$, $\dfrac{3\pi}{2} < A, B < 2\pi$.

3. Prove that $\sin 75^\circ - \sin 15^\circ = \cos 105^\circ + \cos 15^\circ$.

4. If $\cos A = \dfrac{1}{7}$, $\cos B = \dfrac{13}{14}$, prove that $A - B = 60^\circ$. A and B being positive acute angles.

5. Prove that:

(i) $\dfrac{\sin(A-B)}{\sin A \sin B} + \dfrac{\sin(B-C)}{\sin B \sin C} + \dfrac{\sin(C-A)}{\sin C \sin A} = 0$;

(ii) $\sin(n+1)x \sin(n+2)x + \cos(n+1)x \cos(n+2)x = \cos x$;

(iii) $\cos\left(\dfrac{3\pi}{4} + x\right) - \cos\left(\dfrac{3\pi}{4} - x\right) = -\sqrt{2}\sin x$;

(iv) $\sin^2 6x - \sin^2 4x = \sin 2x \sin 10x$;

(v) $\cos^2 2x - \cos^2 6x = \sin 4x \sin 8x$.

6. If $\tan A = \dfrac{a}{a+1}$, $\tan B = \dfrac{1}{2a+1}$, show that $A + B = \dfrac{\pi}{4}$.

7. Prove that

(i) $\tan 13A - \tan 9A - \tan 4A = \tan 13A \tan 9A \tan 4A$;

(ii) $\tan 50^\circ = \tan 40^\circ + 2 \tan 10^\circ$;

(iii) $\tan 75^\circ - \tan 30^\circ - \tan 75^\circ \tan 30^\circ = 1$; (iv) $2 \tan 70^\circ = \tan 80^\circ - \tan 10^\circ$.

8. Prove that

(i) $\dfrac{\cos 29^\circ + \sin 29^\circ}{\cos 29^\circ - \sin 29^\circ} = \tan 74^\circ$; (ii) $\dfrac{\cos 11^\circ + \sin 11^\circ}{\cos 11^\circ - \sin 11^\circ} = \tan 56^\circ$;

(iii) $\dfrac{\cos 10^\circ - \sin 10^\circ}{\cos 10^\circ + \sin 10^\circ} = \tan 35^\circ$; (iv) $\dfrac{\cos 15^\circ - \sin 15^\circ}{\cos 15^\circ + \sin 15^\circ} = \dfrac{1}{\sqrt{3}}$.

9. If $\sin A = \dfrac{8}{17}$ and $\sin B = \dfrac{5}{13}$, find $\sin (A - B)$, where A and B lie in the second quadrant.

10. Prove that

(i) $\sin^2(A + B) - \sin^2(A - B) = \sin 2A \cdot \sin 2B$;

(ii) $\sin^2(15^\circ + A) - \sin^2(15^\circ - A) = \dfrac{1}{2} \sin 2A$;

(iii) $\cos \beta \cdot \cos (2\alpha - \beta) = \cos^2\alpha - \sin^2(\alpha - \beta)$;

(iv) $\cos 2\theta \cos 2\phi + \sin^2(\theta - \phi) - \sin^2(\theta - \phi) = \cos (2\theta + 2\phi)$;

(v) $\cos^2\left(\dfrac{\pi}{4} + x\right) - \sin^2\left(\dfrac{\pi}{4} - x\right)$ is independent of x;

(vi) $\cos\left(\dfrac{\pi}{4} + x\right) + \cos\left(\dfrac{\pi}{4} - x\right) = \sqrt{2} \cos x$.

11. Prove that

(i) $\cot 2\alpha - \tan \alpha = \dfrac{\cos 3\alpha}{\cos \alpha \sin 2\alpha}$; (ii) $\sin^2 A = \cos^2(A - B) + \cos^2 B - 2\cos (A - B) \cos A \cos B$;

(iii) $\tan 2\alpha - \tan \alpha = \tan \alpha \sec 2\alpha$; (iv) $\dfrac{\cos (A + B)}{\cos (A - B)} = \dfrac{1 - \tan A \tan B}{1 + \tan A \tan B}$.

12. Prove that

(i) $\tan\left(\dfrac{\pi}{4} + A\right) \tan\left(\dfrac{\pi}{4} - A\right) = 1$;

(ii) $(\sin A + \sin B)(\sin A - \sin B) = \sin (A + B) \sin (A - B)$;

(iii) $\dfrac{\sin(x - y)}{\sin(x + y)} = \dfrac{\tan x - \tan y}{\tan x + \tan y}$;

(iv) $\cos\left(\dfrac{\pi}{4} - x\right)\cos\left(\dfrac{\pi}{4} - y\right) - \sin\left(\dfrac{\pi}{4} - x\right)\sin\left(\dfrac{\pi}{4} - y\right) = \sin (x + y)$;

(v) $\dfrac{\tan\left(\dfrac{\pi}{4} + x\right)}{\tan\left(\dfrac{\pi}{4} - x\right)} = \dfrac{(1 + \tan x)^2}{(1 - \tan x)^2}$;

(vi) $\dfrac{\cos(\pi + x)\cos(-x)}{\sin(\pi - x)\cos\left(\dfrac{\pi}{2} + x\right)} = \cot^2 x$

(vii) $\cos\left(\dfrac{3\pi}{2} + x\right)\cos(2\pi + x)\left[\cot\left(\dfrac{3\pi}{2} - x\right) + \cot(2\pi + x)\right] = 1.$

13. If $A + B = 225°$, prove that $\dfrac{\cot A}{1 + \cot A} \cdot \dfrac{\cot B}{1 + \cot B} = \dfrac{1}{2}.$

14. Find the value of $\tan(\alpha + \beta)$, given that $\cot\alpha = \dfrac{1}{2}$, $\sec\beta = -\dfrac{5}{3}$, where $\pi < \alpha < \dfrac{3\pi}{2}$ and $\dfrac{\pi}{2} < \beta < \pi$. State the quadrant in which $(\alpha + \beta)$ terminates.

15. If $\tan A = \dfrac{m}{m-1}$ and $\tan B = \dfrac{1}{2m-1}$, prove that $A - B = \dfrac{\pi}{4}.$

16. Find the value of $\sin 163° \cos 347° + \sin 73° \sin 167°$.

LEVEL OF DIFFICULTY B

17. Show that $1 + \tan\theta \tan\dfrac{6}{2} = \tan\theta\cot\dfrac{6}{2} - 1 = \sec\theta.$

18. (i) If $\sin x + \sin y = a$, $\cos x + \cos y = b$, show that

(a) $\cos(x - y) = \dfrac{1}{2}(a^2 + b^2 - 2);$

(b) $\cos(x + y) = \dfrac{b^2 - a^2}{b^2 + a^2}$; (c) $\sin(x + y) = \dfrac{2ab}{a^2 + b^2}$

(ii) If $\tan\theta + \tan\phi = a$ and $\cot\theta + \cot\phi = b$, prove that $\cot(\theta + \phi) = \dfrac{1}{a} - \dfrac{1}{b}.$

19. Show that

$\cos^2 A + \cos^2 B - 2\cos A \cos B \cos(A + B) = \sin^2(A + B).$

20. If $\cot\alpha\cot\beta = 2$, show that $\dfrac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} = \dfrac{1}{3}.$

21. If $\tan\alpha = x + 1$, $\tan\beta = x - 1$, prove that $2\cot(\alpha - \beta) = x^2$.

22. If $\sin(\theta + \phi) = 2\sin(\theta - \phi)$, show that $\tan\theta = 3\tan\phi$.

23. If $\cos(\alpha + \beta) = \dfrac{4}{5}$, $\sin(\alpha - \beta) = \dfrac{5}{13}$ where $0 < \alpha, \beta < \dfrac{\pi}{4}$, prove that $\tan 2\alpha = \dfrac{56}{33}.$

24. If $\sin(\theta + \alpha) = \cos(\theta + \alpha)$, prove that $\tan\theta = \dfrac{1 - \tan\alpha}{1 + \tan\alpha}.$

25. If $\tan(\alpha + \theta) = n\tan(\alpha - \theta)$, show that $(n + 1)\sin 2\theta = (n - 1)\sin 2\alpha$.

26. If $\theta + \phi = \alpha$ and $\sin\theta = k\sin\phi$, prove that

$\tan\theta = \dfrac{k\sin\alpha}{1 + k\cos\alpha}$, $\tan\phi = \dfrac{\sin\alpha}{k + \cos\alpha}.$

27. If $\tan(x + y) = \dfrac{3}{4}$ and $\tan(x - y) = \dfrac{8}{15}$, show that

(i) $\tan 2x = \frac{77}{36}$ (ii) $\tan 2y = \frac{13}{84}$.

28. Prove that $\tan\left(\frac{\pi}{4}+\theta\right)\tan\left(\frac{3\pi}{4}+\theta\right) = -1$.

29. In any quadrilateral $ABCD$, show that
$\cos A \cos B - \cos C \cos D = \sin A \sin B - \sin C \sin D$

30. If $\tan\theta = \frac{a}{b}$ and $\tan\phi = \frac{c}{d}$, prove that $\tan(\theta + \phi) = \frac{ad + bc}{bd - ac}$.

31. If $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -\frac{3}{2}$, prove that
$\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0.$

32. An angle α is divided into two parts such that the ratio of the tangents of the parts is k: 1. If x be the difference of the two parts, prove that $\sin x = \frac{k-1}{k+1}\sin\alpha$.

33. If $\tan x + \tan\left(x+\frac{\pi}{3}\right) + \tan\left(x+\frac{2\pi}{3}\right) = 3$, prove that $\frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} = 1$.

34. If $\cos\alpha + \sin\beta = a$ and $\sin\alpha + \cos\beta = b$, prove that $\sin(\alpha + \beta) = \frac{1}{2}(a^2 + b^2 - 2)$.

35. If $\sin(\alpha + \beta) = 0$ and $\sin(\alpha - \beta) = \frac{1}{2}$, where $0 \le \alpha, \beta \le \frac{\pi}{2}$, find the values of $\tan(\alpha + 2\beta)$ and $\tan(2\alpha + \beta)$.

36. If $\tan(\pi\cos\theta) = \cot(\pi\sin\theta)$, prove that $\cos\left(\theta - \frac{\pi}{4}\right) = \pm\frac{1}{2\sqrt{2}}$.

37. If $A > 0$, $B > 0$ and $A + B = \frac{\pi}{3}$, then find the maximum value of $\tan A \tan B$.

38. Find the maximum value of $5\cos\theta + 3\cos\left(\theta+\frac{\pi}{3}\right) + 3$.

39. Find the value of the expression $\frac{\sin^3 y}{1+\cos y} + \frac{\cos^3 y}{1-\sin y}$.

40. In ΔPQR, $\angle R = \frac{\pi}{2}$. If $\tan\frac{P}{2}$ and $\tan\frac{Q}{2}$ are the roots of equation $ax^2 + bx + c = 0$ $(a \ne 0)$, then show that $a + b = c$.

41. If $3\cos x + 2\cos 3x = \cos y$, $3\sin x + 2\sin 3x = \sin y$, then find the value of $\cos 2x$.

42. Find the maximum value of $1 + \sin\left(\frac{\pi}{2}+\theta\right) + 2\cos\left(\frac{\pi}{4}-\theta\right)$ for real values of θ.

43. If $2\sin\alpha\cdot\cos\beta\cdot\sin\gamma = \sin\beta.\ \sin(\alpha + \gamma)$, then show that $\tan\alpha$, $\tan\beta$, $\tan\gamma$ are in H.P.

44. If $\tan\frac{\alpha}{2}$ and $\tan\frac{\beta}{2}$ are the roots of the equation $8x^2 - 26x + 15 = 0$, then find the value of $\cos(\alpha + \beta)$.

45. If $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\frac{3}{2}$, then show that
$\cos\alpha + \cos\beta + \cos\gamma = 0 = \sin\alpha + \sin\beta + \sin\gamma.$

46. If α and β are the solutions of the equation $a\tan\theta + b\sec\theta = c$, show that
$\tan(\alpha + \beta) = \frac{2ac}{a^2 - c^2}$.

Answers

1. (i) $\frac{\sqrt{3}}{2}$; (ii) $\frac{1}{2}$; (iii) $\frac{1}{2}$; (iv) 0; (v) $\frac{1}{\sqrt{3}}$; (vi) $\frac{\sqrt{3}}{2}$;

(vii) $\frac{\sqrt{3}}{2}$; (viii) $-\frac{\sqrt{3}+1}{2\sqrt{2}}$.

2. (i) $-\cos(A+B+C)$; (ii) 0; (iii) 1; (iv) 1; (v) $-\frac{140}{221}$;

(vi) $-\frac{133}{205}, \frac{187}{205}, \frac{156}{205}, -\frac{84}{205}$; (vii) $\frac{33}{65}, -\frac{16}{65}$.

9. $\frac{-21}{221}$.

10. $-\frac{21}{221}$. **14.** $\frac{2}{11}$, Ist quadrant. **16.** $\frac{1}{2}$. **37.** $\frac{1}{3}$. **38.** 10.

39. $\sqrt{2}\cos\left(\frac{\pi}{4}-y\right)$ or $\sqrt{2}\sin\left(\frac{\pi}{2}+y\right)$. **41.** -1. **42.** 4.**44.** $-\frac{627}{725}$.

HINTS AND SOLUTIONS

1. (viii) $\cos\frac{11\pi}{12} = \cos 165° = \cos(180° - 15°) = -\cos 15° = -\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$

4. Since $\cos A = \frac{1}{7}$ $\therefore$ $\sin^2 A = 1 - \cos^2 A = 1 - \frac{1}{49} = \frac{48}{49}$

$\therefore$ $\sin A = \frac{4\sqrt{3}}{7}$ ($\because$ A is acute) and $\cos B = \frac{13}{14}$

$\therefore$ $\sin B = \sqrt{1-\cos^2 B} = \sqrt{1-\frac{169}{196}} = \sqrt{\frac{27}{196}} = \frac{3\sqrt{3}}{14}$

$\therefore$ $\cos(A-B) = \cos A\cos B + \sin A \sin B$

$$= \frac{1}{7}\cdot\frac{13}{14} + \frac{4\sqrt{3}}{7}\cdot\frac{3\sqrt{3}}{14} = \frac{13+36}{98} = \frac{49}{98} = \frac{1}{2} = \cos 60°$$

$\therefore$ $A - B = 60°$

5. (i) Ist term on L.H.S. $= \frac{\sin(A-B)}{\sin A \sin B} = \frac{\sin A\cos B - \cos A \sin B}{\sin A \sin B}$

$$= \frac{\sin A \cos B}{\sin A \sin B} - \frac{\cos A \sin B}{\sin A \sin B} = \cot B - \cot A$$

Similarly, 2nd term on L.H.S $= \cot C - \cot A$

and 3rd term on L.H.S $= \cot A - \cot B$

$\therefore$ L.H.S $= (\cot B - \cot A) + (\cot C - \cot A) + (\cot A - \cot B) = 0 =$ R.H.S.

(ii) L.H.S $= \cos\left(\frac{3\pi}{4}+x\right) - \cos\left(\frac{3\pi}{4}-x\right)$

Put $\frac{3\pi}{4} + x = \theta$, $\frac{3\pi}{4} - x = \phi$

$\therefore$ L.H.S $= \cos\theta - \cos\phi$

$$= -2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$$

$$= -2\sin\left(\frac{\frac{3\pi}{4}+x+\frac{3\pi}{4}-x}{2}\right)\sin\left(\frac{\left(\frac{3\pi}{4}+x\right)-\left(\frac{3\pi}{4}-x\right)}{2}\right)$$

$$= -2\sin\left(\frac{\frac{3\pi}{2}}{2}\right)\sin\left(\frac{2x}{2}\right)$$

$$= -2\sin\frac{3\pi}{4}\sin x = -2\sin\left(\pi-\frac{\pi}{4}\right)\sin x = -2\sin\frac{\pi}{4}\sin x$$

$$= -2\times\frac{1}{\sqrt{2}}\sin x = -\sqrt{2}\sin x = \text{R.H.S.}$$

(iii) L.H.S. $= \sin^2 6x - \sin^2 4x = \sin(6x+4x)\sin(6x-4x)$

$= \sin 10x \sin 2x =$ R.H.S.

(iv) L.H.S. $= \cos^2 2x - \cos^2 6x = 1 - \sin^2 2x - (1-\sin^2 6x)$

$= \sin^2 6x - \sin 2x = \sin(6x+2x)\sin(6x-2x)$

$= \sin 8x \sin 4x =$ R.H.S.

11. (i) L.H.S $= \dfrac{\cos 2\alpha}{\sin 2\alpha} - \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{\cos 2\alpha\cos\alpha - \sin 2\alpha\sin\alpha}{\sin 2\alpha\cos\alpha}$

$= \dfrac{\cos(2\alpha+\alpha)}{\sin 2\alpha\cos\alpha} = \dfrac{\cos 3\alpha}{\cos\alpha\sin 2\alpha} =$ R.H.S.

(iv) L.H.S $= \dfrac{\cos(A+B)}{\cos(A-B)} = \dfrac{\cos A\cos B - \sin A\sin B}{\cos A\cos B + \sin A\sin B}$

Dividing the numerator and denominator by $\cos A\cos B = \dfrac{1-\tan A\tan B}{1+\tan A\tan B} =$ R.H.S.

13. Here $A + B = 225°$ or $B = 225° - A$

$\therefore$ $\tan B = \tan(225° - A) = \tan(180° + \overline{45° - A}) = \tan(45° - A)$

[$\because \tan(180° + \theta) = \tan\theta$]

$\Rightarrow$ $\tan B = \dfrac{1-\tan A}{1+\tan A}$

$$\text{L.H.S.} = \frac{\cot A}{1+\cot A}\cdot\frac{\cot B}{1+\cot B} = \frac{\frac{1}{\tan A}}{1+\frac{1}{\tan A}}\cdot\frac{\frac{1}{\tan B}}{1+\frac{1}{\tan B}}$$

$$= \frac{1}{\tan A+1}\cdot\frac{1}{\tan B+1} = \frac{1}{1+\tan A}\cdot\frac{1}{1+\tan B}$$

$$= \frac{1}{1+\tan A}\,\frac{1}{1+\frac{1-\tan A}{1+\tan A}} = \frac{1}{1+\tan A}\cdot\frac{1+\tan A}{1+\tan A+1-\tan A}$$

$$= \frac{1}{2} = \text{R.H.S.}$$

14. As β lies in 2nd quadrant, therefore, $\tan\beta$ is negative.

$\therefore \quad \tan\beta = -\sqrt{(\sec^2\beta - 1)} = -\sqrt{\left(\frac{25}{9}-1\right)} = -\frac{4}{3}$

Also $\quad \cot\alpha = \frac{1}{2} \quad \therefore \quad \tan\alpha = 2$

$$\therefore \quad \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} = \frac{2-\frac{4}{3}}{1-(2)\left(-\frac{4}{3}\right)} = \frac{2}{11}$$

Now, tangent is positive in I and III quadrants.

Also, $\alpha > \pi$, and $\beta > \frac{\pi}{2} \Rightarrow \alpha + \beta > \left(\pi + \frac{\pi}{2}\right) = \frac{3\pi}{2}$

i.e., beyond III quadrant. So, it must lie in the Ist quadrant.

16. $163° = 180° - 17°$, $347° = 360° - 13°$

$73° = 90° - 17°$, $167° = 180° - 13°$

Given expression $= \sin 17° \cos 13° + \cos 17° \sin 13° = \sin(17° + 13°) = \sin 30° = \frac{1}{2}$

17. $$1 + \tan\theta\tan\frac{\theta}{2} = 1 + \frac{\sin\theta}{\cos\theta} \times \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}} = \frac{\cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2}}{\cos\theta\cos\frac{\theta}{2}}$$

$$= \frac{\cos\left(\theta - \frac{\theta}{2}\right)}{\cos\theta\cos\frac{\theta}{2}} = \frac{1}{\cos\theta} = \sec\theta$$

$$\tan\theta\cot\frac{\theta}{2} - 1 = \frac{\sin\theta\cos\frac{\theta}{2}}{\cos\theta\sin\frac{\theta}{2}} - 1 = \frac{\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}}{\cos\theta\sin\frac{\theta}{2}}$$

$$= \frac{\sin\left(\theta - \frac{\theta}{2}\right)}{\cos\theta\sin\frac{\theta}{2}} = \frac{1}{\cos\theta} = \sec\theta$$

$$1 + \tan\theta\tan\frac{\theta}{2} = \tan\theta\cot\frac{\theta}{2} - 1 = \sec\theta.$$

18. (i) (a) Given $\quad a = \sin x + \sin y$

$b = \cos x + \cos y$

Squaring and adding, we get

$$a^2 + b^2 = 2 + 2(\cos x\cos y + \sin x\sin y) = 2 + 2\cos(x-y)$$

$$\therefore \quad \cos(x-y) = \frac{1}{2}(a^2 + b^2 - 2)$$

(ii) $\cot(\theta+\phi) = \dfrac{1}{\tan(\theta+\phi)} = \dfrac{1-\tan\theta\tan\phi}{\tan\theta+\tan\phi} = \dfrac{1}{\tan\theta+\tan\phi} - \dfrac{\tan\theta\tan\phi}{\tan\theta+\tan\phi}$

$$= \frac{1}{\tan\theta+\tan\phi} - \frac{1}{\left(\dfrac{1}{\tan\theta}+\dfrac{1}{\tan\phi}\right)}$$

$$= \frac{1}{(\tan\theta+\tan\phi)} - \frac{1}{(\cot\theta+\cot\phi)} = \frac{1}{a} - \frac{1}{b}$$

19. L.H.S. $= \cos^2 A + \cos^2 B - 2\cos A\cos B\cos(A+B)$

$= \cos^2 A + (1 - \sin^2 B) - 2\cos A\cos B\cos(A+B)$

$= 1 + (\cos^2 A - \sin^2 B) - 2\cos A\cos B\cos(A+B)$

$= 1 + \cos(A+B)\cos(A-B) - 2\cos A\cos B\cos(A+B)$

$= 1 + \cos(A+B)[\cos(A-B) - 2\cos A\cos B]$

$= 1 + \cos(A+B)[\cos A\cos B + \sin A\sin B - 2\cos A\cos B]$

$= 1 + \cos(A+B)[\sin A\sin B - \cos A\cos B]$

$= 1 + \cos(A+B)[-\cos(A+B)] = 1 - \cos^2(A+B) = \sin^2(A+B) =$ R.H.S.

20. $\cot\alpha\cot\beta = 2 \Rightarrow \dfrac{\cot\alpha}{\sin\alpha}\cdot\dfrac{\cos\beta}{\sin\beta} = \dfrac{2}{1} \Rightarrow \dfrac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta} = \dfrac{2}{1}$

By componendo and dividendo,

$$\frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\cos\beta + \sin\alpha\sin\beta} = \frac{2-1}{2+1} \Rightarrow \frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)} = \frac{1}{3}$$

21. $\tan\alpha = x+1,\ \tan\beta = x-1$ ...(1)

$$\cot(\alpha-\beta) = \frac{1}{\tan(\alpha-\beta)} = \frac{1}{\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}} = \frac{1+\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}$$

$$= \frac{1+(x+1)(x-1)}{(x+1)-(x-1)} = \frac{x^2}{2}$$ [Using (1)]

$\therefore\quad 2\cot(\alpha-\beta) = x^2$

23. Since α, and β lie between 0 and $\pi/4$, therefore, $-\pi/4 < \alpha - \beta < \pi/4$ and $0 < \alpha+\beta < \pi/2$. Consequently, $\cos(\alpha-\beta)$ and $\sin(\alpha+\beta)$ are positive.

Now, $\sin(\alpha+\beta) = \sqrt{1-\cos^2(\alpha+\beta)} \Rightarrow \sin(\alpha+\beta) = \sqrt{1-\dfrac{16}{25}} = \dfrac{3}{5}$

and $\cos(\alpha-\beta) = \sqrt{1-\sin^2(\alpha-\beta)} \Rightarrow \cos(\alpha-\beta) = \sqrt{1-\dfrac{25}{169}} = \dfrac{12}{13}$

$\therefore\quad \tan(\alpha+\beta) = \dfrac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)} = \dfrac{3/5}{4/5} = \dfrac{3}{4}$

and $\tan(\alpha-\beta) = \dfrac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)} = \dfrac{5}{12}$

$\Rightarrow \quad \tan 2\alpha = \tan[(\alpha+\beta)+(\alpha-\beta)] = \dfrac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\tan(\alpha-\beta)} = \dfrac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\times\frac{5}{12}}$

$\Rightarrow \quad \tan 2\alpha = \dfrac{56}{33}$

25. $\tan(\alpha+\theta) = n\tan(\alpha-\theta)$

$\Rightarrow \quad \dfrac{\tan(\alpha+\theta)}{\tan(\alpha-\theta)} = \dfrac{n}{1} \Rightarrow \dfrac{\tan(\alpha+\theta)+\tan(\alpha-\theta)}{\tan(\alpha+\theta)-\tan(\alpha-\theta)} = \dfrac{n+1}{n-1}$

[Applying componendo and dividendo]

$\Rightarrow \quad \dfrac{\sin(\alpha+\theta)\cos(\alpha-\theta)+\cos(\alpha+\theta)\sin(\alpha-\theta)}{\sin(\alpha+\theta)\cos(\alpha-\theta)-\cos(\alpha+\theta)\sin(\alpha-\theta)} = \dfrac{n+1}{n-1}$

$\Rightarrow \quad \dfrac{\sin[(\alpha+\theta)+(\alpha-\theta)]}{\sin[(\alpha+\theta)-(\alpha-\theta)]} = \dfrac{n+1}{n-1} \Rightarrow \dfrac{\sin 2\alpha}{\sin 2\theta} = \dfrac{n+1}{n-1}$

$\Rightarrow \quad (n+1)\sin 2\theta = (n-1)\sin 2\alpha.$

26. Here $\theta+\phi = \alpha$, $\sin\theta = k\sin\phi$ ...(1)

$\Rightarrow \quad \sin\theta = k\sin\phi = k\sin(\alpha-\theta)$ $[\because \theta+\phi=\alpha, \therefore \phi = \alpha-\theta]$

$= k(\sin\alpha\cos\theta - \cos\alpha\sin\theta) = k\sin\alpha\cos\theta - k\cos\alpha\sin\theta$

Dividing both sides by $\cos\phi$,

$\tan\theta = k\sin\alpha - k\cos\alpha\tan\theta \Rightarrow (1+k\cos\alpha)\tan\theta = k\sin\alpha$

$\therefore \quad \tan\theta = \dfrac{k\sin\alpha}{1+k\cos\alpha}$

Again, $\sin\theta = k\sin\phi$ $[\because \theta+\phi=\alpha, \therefore \theta=\alpha-\phi]$

$\sin(\alpha-\phi) = k\sin\phi \Rightarrow \sin\alpha\cos\phi - \cos\alpha\sin\phi = k\sin\phi$

Dividing both sides by $\cos\phi$, we get

$\sin\alpha - \cos\alpha\tan\phi = k\tan\phi$

$\Rightarrow \quad (k+\cos\alpha)\tan\phi = \sin\alpha \quad \therefore \quad \tan\phi = \dfrac{\sin\alpha}{k+\cos\alpha}$

27. (i) $\tan 2x = \tan[(x+y)+(x-y)] = \dfrac{\tan(x+y)-\tan(x-y)}{1+\tan(x+y)\tan(x-y)}$

$= \dfrac{\frac{3}{4}+\frac{8}{15}}{1-\frac{3}{4}\times\frac{8}{15}} = \dfrac{45+32}{60}\times\dfrac{60}{60-24} = \dfrac{77}{36}$

(ii) $\tan 2y = \tan[(x+y)-(x-y)] = \dfrac{\tan(x+y)-\tan(x-y)}{1+\tan(x+y)\tan(x-y)}$

$= \dfrac{\frac{3}{4}-\frac{8}{15}}{1+\frac{3}{4}\times\frac{8}{15}} = \dfrac{45-32}{60}\times\dfrac{60}{60+24} = \dfrac{13}{84}$

28. $\tan\left(\dfrac{\pi}{4}+\theta\right)\tan\left(\dfrac{3\pi}{4}+\theta\right) = \tan\left(\dfrac{\pi}{4}+\theta\right)\tan\left[\dfrac{\pi}{2}+\left(\dfrac{\pi}{4}+\theta\right)\right]$

$= -\tan\left(\dfrac{\pi}{4}+\theta\right)\cot\left(\dfrac{\pi}{4}+\theta\right)$ $\left[\because \tan\left(\dfrac{\pi}{4}+\theta\right) = -\cot\theta\right]$

$= -1 = $ R.H.S.

29. In any quadrilateral $ABCD$

$$A + B + C + D = 2\pi$$

$\therefore \quad A + B = 2\pi - (C + D)$

$\therefore \quad \cos(A + B) = \cos[2\pi - (C + D)] = \cos(C + D)$

$\therefore \quad \cos A \cos B - \sin A \sin B = \cos C \cos D - \sin C \sin D$

$\Rightarrow \quad \cos A \cos B - \cos C \cos D = \sin A \sin B - \sin C \sin D.$

31. $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = -\frac{3}{2}$

$\Rightarrow \quad 2\cos\alpha\cos\beta + 2\sin\alpha\sin\beta + 2\cos\beta\cos\gamma + 2\sin\beta\sin\gamma + 2\cos\gamma\cos\alpha + 2\sin\gamma\sin\alpha = -3$

$\Rightarrow \quad (2\cos\alpha\cos\beta + 2\cos\beta\cos\gamma + 2\cos\gamma\cos\alpha) + (2\sin\alpha\sin\beta + 2\sin\beta\sin\gamma + 2\sin\gamma\sin\alpha) + 3 = 0$

$\Rightarrow \quad (2\cos\alpha\cos\beta + 2\cos\beta\cos\gamma + 2\cos\gamma\cos\alpha) + (2\sin\alpha\sin\beta + 2\sin\beta\sin\gamma + 2\sin\gamma\sin\alpha + (\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) + (\cos^2\gamma + \sin^2\gamma) = 0$

$\Rightarrow \quad (\cos^2\alpha\ \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta + 2\cos\beta\cos\gamma + 2\cos\gamma\cos\alpha) + (\sin^2\alpha + \sin^2\beta + \sin^2\gamma + 2\sin\alpha\sin\beta + 2\sin\beta\sin\gamma + 2\sin\gamma\sin\alpha) = 0$

$\Rightarrow \quad (\cos\alpha + \cos\beta + \cos\gamma)^2 + (\sin\alpha + \sin\beta + \sin\gamma)^2 = 0$

$\Rightarrow \quad \cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$

32. Let the two parts be θ and ϕ, $\theta > \phi$, then

$$\theta + \phi = \alpha \text{ and } \theta - \phi = x \qquad ...(1)$$

Also $\dfrac{\tan\theta}{\tan\phi} = \dfrac{k}{1} \Rightarrow \dfrac{\sin\theta\cos\phi}{\cos\theta\sin\phi} = \dfrac{k}{1}$

By componendo and dividendo, we have

$$\frac{\sin\theta\cos\phi - \cos\theta\sin\phi}{\sin\theta\cos\phi + \cos\theta\sin\phi} = \frac{k-1}{k+1} \quad \text{or} \quad \frac{\sin(\theta - \phi)}{\sin(\theta + \phi)} = \frac{k-1}{k+1}$$

or $\quad \dfrac{\sin x}{\sin\alpha} = \dfrac{k-1}{k+1}$ [Using (1)]

$\therefore \quad \sin x = \dfrac{k-1}{k+1}\sin\alpha$

34. Square and add.

36. $\tan(\pi\cos\theta) = \cot(\pi\sin\theta) \Rightarrow \dfrac{\sin(\pi\cos\theta)}{\cos(\pi\cos\theta)} = \dfrac{\cos(\pi\sin\theta)}{\sin(\pi\sin\theta)}$

$\Rightarrow \quad \sin(\pi\cos\theta)\sin(\pi\sin\theta) = \cos(\pi\sin\theta)\cos(\pi\cos\theta)$

$\Rightarrow \quad \cos(\pi\cos\theta)\cos(\pi\sin\theta) - \sin(\pi\cos\theta)\sin(\pi\sin\theta) = 0$

$\Rightarrow \quad \cos(\pi\cos\theta + \pi\sin\theta) = 0 \Rightarrow \pi\cos\theta + \pi\sin\theta = \pm\pi/2$ $\left[\because \cos\left(\pm\frac{\pi}{2}\right) = 0\right]$

$\Rightarrow \quad \cos\theta + \sin\theta = \pm\frac{1}{2}$

$\Rightarrow \quad \dfrac{1}{\sqrt{2}}\cos\theta + \dfrac{1}{\sqrt{2}}\sin\theta = \pm\dfrac{1}{2\sqrt{2}}$ [Multiplying both sides by $\frac{1}{\sqrt{2}}$]

$\Rightarrow \quad \cos\theta\cos\dfrac{\pi}{4} + \sin\theta\sin\dfrac{\pi}{4} = \pm\dfrac{1}{2\sqrt{2}} \Rightarrow \cos\left(\theta - \dfrac{\pi}{4}\right) = \pm\dfrac{1}{2\sqrt{2}}$

37. Let $y = \tan A \tan B$

$\because$ A.M. $\geq$ G.M. $\therefore \dfrac{\tan A + \tan B}{2} \geq \sqrt{\tan A \tan B}$

$\Rightarrow \dfrac{\sqrt{3}\,(1 - \tan A \tan B)}{2} \geq \sqrt{\tan A \tan B}$

$$\left[\because \tan(A+B) = \tan\frac{\pi}{3}, \therefore \tan A + \tan B = \sqrt{3}[1 - \tan A \tan B]\right]$$

$\Rightarrow 3(1 - y)^2 \geq 4y \Rightarrow 3y^2 - 10y + 3 \geq 0 \Rightarrow y \leq \dfrac{1}{3}$ or $y \geq 3$

$\Rightarrow y \leq \dfrac{1}{3}$ $\quad \left[\because 0 < A < \dfrac{\pi}{3}, 0 < B < \dfrac{\pi}{3} \;\therefore \tan A \tan B < 3\right]$

Hence max. value of $\tan A \tan B = \dfrac{1}{3}$

38. Given expression $= 5\cos\theta + 3\left(\dfrac{1}{2}\cos\theta - \dfrac{\sqrt{3}}{2}\sin\theta\right) + 3 = \dfrac{13}{2}\cos\theta - \dfrac{3\sqrt{3}}{2}\sin\theta + 3$ ...(1)

But $\sqrt{\left(\dfrac{13}{2}\right)^2 + \left(\dfrac{3\sqrt{3}}{2}\right)^2} = \sqrt{\dfrac{169+27}{4}} = \sqrt{\dfrac{196}{4}} = 7$

Max. and min. values of $\dfrac{13}{2}\cos\theta - \dfrac{3\sqrt{3}}{2}\sin\theta$ are 7 and -7.

Max. value of given expression $= 7 + 3 = 10$

39. Given expression $= \dfrac{\sin^3 y(1 - \cos y)}{\sin^2 y} + \dfrac{\cos^3 y(1 + \sin y)}{\cos^2 y}$

$= \sin y - \sin y \cos y + \cos y + \sin y \cos y$

$= \sin y + \cos y = \sqrt{2}\cos\left(\dfrac{\pi}{4} - y\right)$ or $\sqrt{2}\sin\left(\dfrac{\pi}{4} + y\right)$

40. $\tan\dfrac{P}{2} + \tan\dfrac{Q}{2} = -\dfrac{b}{a}$, $\tan\dfrac{P}{2}\tan\dfrac{Q}{2} = \dfrac{c}{a}$

$\angle R = \dfrac{\pi}{2} \quad \therefore \quad \dfrac{P}{2} + \dfrac{Q}{2} = \dfrac{\pi}{4}$

$\therefore \dfrac{\tan\frac{P}{2} + \tan\frac{Q}{2}}{1 - \tan\frac{P}{2}\tan\frac{Q}{2}} = \tan\dfrac{\pi}{4} = 1 \Rightarrow \dfrac{-\frac{b}{a}}{1 - \frac{c}{a}} = a \Rightarrow b = c - a \Rightarrow a + b = c$

41. $\sin^2 y + \cos^2 y = 9 + 4 + 12\cos(3x - x) \Rightarrow \cos 2x = -1$

42. $1 + \sin\left(\dfrac{\pi}{4} + \theta\right) + 2\cos\left(\dfrac{\pi}{4} - \theta\right) = 1 + \dfrac{1}{\sqrt{2}}(\cos\theta + \sin\theta) + \sqrt{2}(\cos\theta + \sin\theta)$

$= 1 + \left(\dfrac{1}{\sqrt{2}} + \sqrt{2}\right)\cdot(\cos\theta + \sin\theta)$

$= 1 + \left(\dfrac{1}{\sqrt{2}} + \sqrt{2}\right)\cdot\sqrt{2}\cos\left(\theta - \dfrac{\pi}{4}\right)$

Maximum value $= 1 + \left(\dfrac{1}{\sqrt{2}} + \sqrt{2}\right)\sqrt{2} = 4$

43. $2 \sin \alpha \cdot \cos \beta \cdot \sin \gamma = \sin \alpha \cdot \sin \beta \cdot \cos \gamma + \sin \beta \cdot \sin \gamma \cdot \cos \alpha$

Dividing by $\sin \alpha \cdot \sin \beta \cdot \sin \gamma$, $2 \cot \beta = \cot \gamma + \cot \alpha$.

44. Use $\cos(\alpha + \beta) = \dfrac{1 - \tan^2 \frac{\alpha+\beta}{2}}{1 + \tan^2 \frac{\alpha+\beta}{2}}$, $\tan \dfrac{\alpha+\beta}{2} = \dfrac{\tan\frac{\alpha}{2} + \tan\frac{\beta}{2}}{1 - \tan\frac{\alpha}{2}\tan\frac{\beta}{2}}$

and $\tan \dfrac{\alpha}{2} + \tan \dfrac{\beta}{2} = \dfrac{13}{4}$, $\tan \dfrac{\alpha}{2} \tan \dfrac{\beta}{2} = \dfrac{15}{8}$.

45. Given, $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\dfrac{3}{2}$

$\Rightarrow \quad 3 + 2\cos(\beta - \gamma) + 2\cos(\gamma - \alpha) + 2\cos(\alpha - \beta) = 0$

$\Rightarrow \quad 3 + 2(\cos\beta \cos\gamma + \sin\beta \sin\gamma) + 2(\cos\gamma \cos\alpha + \sin\gamma \sin\alpha)$
$+ 2(\cos\alpha \cos\beta + \sin\alpha \sin\beta) = 0$

$\Rightarrow \quad (\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta) + (\cos^2\gamma + \sin^2\gamma) + 2(\cos\beta \cos\gamma + \sin\beta \sin\gamma)$
$+ 2(\cos\gamma \cos\alpha + \sin\gamma \sin\alpha) + 2(\cos\alpha \cos\beta + \sin\alpha \sin\beta) = 0$

$\Rightarrow \quad (\cos^2\alpha + \cos^2\beta + \cos^2\gamma + 2\cos\alpha\cos\beta + 2\cos\beta\cos\gamma + 2\cos\gamma\cos\alpha)$
$+ (\sin^2\alpha + \sin^2\beta + \sin^2\gamma + 2\sin\alpha\sin\beta + 2\sin\beta\sin\gamma + 2\sin\gamma\sin\alpha) = 0$

$\Rightarrow \quad (\cos\alpha + \cos\beta + \cos\gamma)^2 + (\sin\alpha + \sin\beta + \sin\gamma)^2 = 0$

which is possible only when

$\cos\alpha + \cos\beta + \cos\gamma = 0$ and $\sin\alpha + \sin\beta + \sin\gamma = 0$.

TRANSFORMATION OF A PRODUCT INTO A SUM OR DIFFERENCE

To prove that:

(i) $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$;

(ii) $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$;

(iii) $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$;

(iv) $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$.

Proof. (i) $\sin(A + B) + \sin(A - B) = (\sin A \cos B + \cos A \sin B) + (\sin A \cos B - \cos A \sin B)$
$= 2 \sin A \cos B$

$\therefore \quad 2 \sin A \cos B = \sin(A + B) + \sin(A - B)$

(ii) $\sin(A + B) - \sin(A - B) = (\sin A \cos B + \cos A \sin B) - (\sin A \cos B - \cos A \sin B)$
$= 2 \cos A \sin B$

$\therefore \quad 2 \cos A \sin B = \sin(A + B) - \sin(A - B)$

(iii) $\cos(A + B) + \cos(A - B) = (\cos A \cos B - \sin A \sin B) + (\cos A \cos B + \sin A \sin B)$
$= 2 \cos A \cos B$

$\therefore \quad 2 \cos A \cos B = \cos(A + B) + \cos(A - B)$

(iv) $\cos(A - B) - \cos(A + B) = (\cos A \cos B + \sin A \sin B) - (\cos A \cos B - \sin A \sin B)$
$= 2 \sin A \sin B$

$\therefore \quad 2 \sin A \sin B = \cos(A - B) - \cos(A + B)$

TRANSFORMATION OF A SUM OR DIFFERENCE INTO A PRODUCT

To prove that:

(i) $\sin C + \sin D = 2 \sin \dfrac{C + D}{2} \cos \dfrac{C - D}{2}$;

(ii) $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$;

(iii) $\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$;

(iv) $\cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$.

Proof. Let $A + B = C$ and $A - B = D$. ...(1)

Then by addition and subtraction,

$$2A = C + D \text{ and } 2B = C - D \quad \text{or} \quad A = \frac{C+D}{2} \text{ and } B = \frac{C-D}{2} \qquad ...(2)$$

Then

(i) $\sin C + \sin D = \sin(A+B) + \sin(A-B)$ [From (1)]

$= (\sin A\cos B + \cos A\sin B) + (\sin A\cos B - \cos A\sin B)$

$= 2\sin A\cos B = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$ [From (2)]

$\therefore \quad \sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$

(ii) $\sin C - \sin D = \sin(A+B) - \sin(A-B)$ [From (1)]

$= (\sin A\cos B + \cos A\sin B) - (\sin A\cos B - \cos A\sin B)$

$= 2\cos A\sin B = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$ [From (2)]

$\therefore \quad \sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$

(iii) $\cos C + \cos D = \cos(A+B) + \cos(A-B)$ [From (1)]

$= (\cos A\cos B - \sin A\sin B) + (\cos A\cos B + \sin A\sin B)$

$= 2\cos A\cos B = 2\cos\frac{C+D}{2}\cos\left(\frac{C-D}{2}\right)$ [From (2)]

$\therefore \quad \cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$

(iv) $\cos C - \cos D = \cos(A+B) - \cos(A-B)$ [From (1)]

$= (\cos A\cos B - \sin A\sin B) - (\cos A\cos B + \sin A\sin B)$

$= -2\sin A\sin B = 2\sin A(-\sin B)$

$= 2\sin A\sin(-B)$ [$\because -\sin\theta = \sin(-\theta)$]

$= 2\sin\frac{C+D}{2}\sin\left[-\left(\frac{C-D}{2}\right)\right]$ [From (2)]

$= 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$

$\therefore \quad \cos C - \cos D = 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D-C}{2}\right)$

$\therefore \quad \cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}$

CAUTION

- The above formulae are popularly known as *C D formulae.*
- The '*C D* formulae' are used only when both the terms on L.H.S. are sines or cosines.
- The result of the formula (iv) should be carefully noted. The last factor on the R.H.S. is $\sin \frac{D-C}{2}$ and not $\sin \frac{C-D}{2}$

Example 1. *Express the following as a sum or difference:*

(i) $2 \sin \theta \cos 3\theta$; (ii) $2 \sin \frac{5\theta}{2} \cos \frac{7\theta}{2}$;

(iii) $2 \cos (\alpha + \beta) \cos (\alpha - \beta)$; (iv) $\cos 22\frac{1}{2}^\circ \cos 67\frac{1}{2}^\circ$.

Solution: (i) $2 \sin \theta \cos 3\theta = \sin (\theta + 3\theta) + \sin (\theta - 3\theta)$

$[\because 2 \sin A \cos B = \sin (A + B) + \sin (A - B)]$

$= \sin 4\theta + \sin (-2\theta) = \sin 4\theta - \sin 2\theta.$

(ii) $2 \sin \frac{5\theta}{2} \cos \frac{7\theta}{2} = \sin \left(\frac{5\theta}{2} + \frac{7\theta}{2}\right) + \sin \left(\frac{5\theta}{2} - \frac{7\theta}{2}\right)$

$[\because 2 \sin A \cos B = \sin (A + B) + \sin (A - B)]$

$= \sin 6\theta + \sin (-\theta) = \sin 6\theta - \sin \theta.$ $[\because \sin (-\theta) = -\sin \theta]$.

(iii) $2 \cos (\alpha + \beta) \cos (\alpha - \beta) = \cos [(\alpha + \beta) + (\alpha - \beta)] + \cos [(\alpha + \beta) - (\alpha - \beta)]$

$[\because 2 \cos A \cos B = \cos (A + B) + \cos (A - B)]$

$= \cos 2\alpha + \cos 2\beta.$

(iv) $\cos 22\frac{1}{2}^\circ \cos 67\frac{1}{2}^\circ = \frac{1}{2}\left[2 \cos 22\frac{1}{2}^\circ \cos 67\frac{1}{2}^\circ\right]$

$= \frac{1}{2}\left[\cos\left(22\frac{1}{2}^\circ + 67\frac{1}{2}^\circ\right) + \cos\left(22\frac{1}{2}^\circ - 67\frac{1}{2}^\circ\right)\right]$

$[\because 2 \cos A \cos B = \cos (A + B) + \cos (A - B)]$

$= \frac{1}{2}[\cos 90^\circ + \cos (-45^\circ)] = \frac{1}{2}[\cos 90^\circ + \cos 45^\circ]$

$[\because \cos (-\theta) = \cos \theta]$

$= \frac{1}{2}\left[0 + \frac{1}{\sqrt{2}}\right] = \frac{1}{2\sqrt{2}}.$ $\left[\because \cos 90^\circ = 0, \ \cos 45^\circ = \frac{1}{\sqrt{2}}\right]$

Example 2. *Express the following in the form of a product:*

(i) $\sin 14\theta + \sin 2\theta$; (ii) $\cos 10^\circ - \cos 50^\circ$

(iii) $\cos (\theta + \phi) + \cos (\theta - \phi)$ (iv) $\sin\left(\frac{\alpha+\beta}{2}\right) - \sin\left(\frac{\alpha-\beta}{2}\right)$.

Solution: (i) $\sin 14\theta + \sin 2\theta = 2 \sin \frac{14\theta + 2\theta}{2} \cos \frac{14\theta - 2\theta}{2}$

$\left[\because \sin C + \sin D = 2 \sin \frac{C + D}{2} \cos \frac{C - D}{2}\right]$

$= 2 \sin 8\theta \cos 6\theta$

(ii) $\cos 10° - \cos 50°$

$$= 2\sin\frac{10°+50°}{2}\sin\frac{50°-10°}{2} \qquad \left[\because \cos C - \cos D = 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}\right]$$

$$= 2\sin 30°\sin 20° = 2\cdot\frac{1}{2}\cdot\sin 20° = \sin 20°. \qquad \left[\because \sin 30° = \frac{1}{2}\right]$$

(iii) $\cos(\theta+\phi) + \cos(\theta-\phi) = 2\cos\frac{(\theta+\phi)+(\theta-\phi)}{2}\cos\frac{(\theta+\phi)-(\theta-\phi)}{2}$

$$= 2\cos\theta\cos\phi.$$

(iv) $\sin\frac{\alpha+\beta}{2} - \sin\frac{\alpha-\beta}{2} = 2\cos\frac{\frac{\alpha+\beta}{2}+\frac{\alpha-\beta}{2}}{2}\sin\frac{\frac{\alpha+\beta}{2}-\frac{\alpha-\beta}{2}}{2} = 2\cos\frac{\alpha}{2}\sin\frac{\beta}{2}.$

Example 3. *Prove that:* $\cos 20°\cos 40°\cos 60°\cos 80° = \frac{1}{16}$

Solution: L.H.S. $= \cos 20°\cos 40°\cos 60°\cos 80°$

$$= \frac{1}{2}\cos 20°\cos 40°\cos 80° \qquad \left[\because \cos 60° = \frac{1}{2}\right]$$

$$= \frac{1}{4}(2\cos 20°\cos 40°)\cos 80°$$

$$= \frac{1}{4}[\cos(20°+40°) + \cos(20°-40°)]\cos 80°.$$

$$[\because\ 2\cos A\cos B = \cos(A+B) + \cos(A-B)]$$

$$= \frac{1}{4}(\cos 60° + \cos 20°)\cos 80° \qquad [\because\ \cos(-20°) = \cos 20°]$$

$$= \frac{1}{4}\left(\frac{1}{2}\cos 80° + \cos 20°\cos 80°\right) \qquad \left[\because \cos 60° = \frac{1}{2}\right]$$

$$= \frac{1}{8}(\cos 80° + 2\cos 20°\cos 80°)$$

$$= \frac{1}{8}[\cos 80° + \cos(20°+80°) + \cos(20°-80°)]$$

$$[\because\ 2\cos A\cos B = \cos(A+B) + \cos(A-B)]$$

$$= \frac{1}{8}[\cos 80° + \cos 100° + \cos(-60°)] = \frac{1}{8}\left[\cos 80° - \cos 80° + \frac{1}{2}\right]$$

$$= \frac{1}{8}\cdot\frac{1}{2} = \frac{1}{16} = \text{R.H.S.}$$

$$\left[\because \cos 100° = \cos(180°-80°) = -\cos 80° \text{ and } \cos(-60°) = \cos 60° = \frac{1}{2}\right]$$

Example 4. *Prove that* $4\sin\theta\sin\left(\theta+\frac{\pi}{3}\right)\sin\left(\theta+\frac{2\pi}{3}\right) = \sin 3\theta.$

Solution: $4\sin\theta\sin\left(\theta+\dfrac{\pi}{3}\right)\sin\left(\theta+\dfrac{2\pi}{3}\right) = 2\sin\theta\cdot 2\sin\left(\theta+\dfrac{2\pi}{3}\right)\sin\left(\theta+\dfrac{\pi}{3}\right)$

$$= 2\sin\theta\left[\cos\left(\theta+\frac{2\pi}{3}-\theta-\frac{\pi}{3}\right)-\cos\left(\theta+\frac{2\pi}{3}+\theta+\frac{\pi}{3}\right)\right]$$

$$= 2\sin\theta\left[\cos\frac{\pi}{3}-\cos(\pi+2\theta)\right] = 2\sin\theta\left[\frac{1}{2}-(-\cos2\theta)\right]$$

$$= \sin\theta + 2\cos2\theta\sin\theta = \sin\theta + \sin(2\theta+\theta) - \sin(2\theta-\theta)$$

$$= \sin\theta + \sin3\theta - \sin\theta = \sin3\theta.$$

Example 5. *Prove that:*

$$\cos^2 A + \cos^2 B - 2\cos A\cos B\cos(A+B) = \sin^2(A+B).$$

Solution: L.H.S. $= \cos^2 A + \cos^2 B - 2\cos A\cos B\cos(A+B)$

$= \cos^2 A + \cos^2 B - [\cos(A+B)+\cos(A-B)]\cos(A+B)$

$[\because 2\cos A\cos B = \cos(A+B)+\cos(A-B)]$

$= \cos^2 A + \cos^2 B - \cos^2(A+B) - \cos(A-B)\cos(A+B)$

$= \cos^2 A + \cos^2 B - \cos^2(A+B) - (\cos^2 A - \sin^2 B)$

$[\because \cos(A+B)\cos(A-B) = \cos^2 A - \sin^2 B]$

$= (\sin^2 B + \cos^2 B) - \cos^2(A+B) = 1 - \cos^2(A+B)$

$[\because \sin^2 B + \cos^2 B = 1]$

Prove: $= \sin^2(A+B) = $ R.H.S.

Example 6. $2\cos\dfrac{\pi}{13}\cos\dfrac{9\pi}{13}+\cos\dfrac{3\pi}{13}+\cos\dfrac{5\pi}{13}=0$

Solution: $2\cos\dfrac{\pi}{13}\cos\dfrac{9\pi}{13}+\cos\dfrac{3\pi}{13}+\cos\dfrac{5\pi}{13}$

$$= \cos\left(\frac{\pi}{13}+\frac{9\pi}{13}\right)+\cos\left(\frac{\pi}{13}-\frac{9\pi}{13}\right)+\cos\frac{3\pi}{13}+\cos\frac{5\pi}{13}$$

$$= \cos\frac{10\pi}{13}+\cos\left(-\frac{8\pi}{13}\right)+\cos\left(\pi-\frac{10\pi}{13}\right)+\cos\left(\pi-\frac{8\pi}{13}\right)$$

$$= \cos\frac{10\pi}{13}+\cos\frac{8\pi}{13}-\cos\frac{10\pi}{13}-\cos\frac{8\pi}{13}=0$$

Example 7. *Prove that*

$$\cos2\theta\cos\frac{\theta}{2}-\cos3\theta\cos\frac{9\theta}{2}=\sin5\theta\sin\frac{5\theta}{2}.$$

Solution: $\cos2\theta\cos\dfrac{\theta}{2}-\cos3\theta\cos\dfrac{9\theta}{2} = \dfrac{1}{2}\left(2\cos2\theta\cos\dfrac{\theta}{2}-2\cos3\theta\cos\dfrac{9\theta}{2}\right)$

$$= \frac{1}{2}\left[\left\{\cos\left(2\theta+\frac{\theta}{2}\right)+\cos\left(2\theta-\frac{\theta}{2}\right)\right\}-\left\{\cos\left(3\theta+\frac{9\theta}{2}\right)+\cos\left(3\theta-\frac{9\theta}{2}\right)\right\}\right]$$

$$= \frac{1}{2}\left[\cos\frac{5\theta}{2}+\cos\frac{3\theta}{2}-\cos\frac{15\theta}{2}-\cos\left(-\frac{3\theta}{2}\right)\right]$$

$$= \frac{1}{2}\left[\cos\frac{5\theta}{2} + \cos\frac{3\theta}{2} - \cos\frac{15\theta}{2} - \cos\frac{3\theta}{2}\right] \qquad [\because \cos(-A) = \cos A]$$

$$= \frac{1}{2}\left[\cos\frac{5\theta}{2} - \cos\frac{15\theta}{2}\right] = \frac{1}{2}\times 2\sin\frac{\frac{5\theta}{2}+\frac{15\theta}{2}}{2}\sin\frac{\frac{15\theta}{2}-\frac{5\theta}{2}}{2} = \sin 5\theta \sin\frac{5\theta}{2}$$

Example 8. *Prove that:* $\sin 10° \sin 50° \sin 70° = \frac{1}{8}$.

Solution: L.H.S. $= \sin 10° \sin 50° \sin 70° = \frac{1}{2}\sin 10°(2\sin 70° \sin 50°)$

$$= \frac{1}{2}\sin 10°[\cos(70° - 50°) - \cos(70° + 50°)]$$

$$[\because 2\sin A \sin B = \cos(A - B) - \cos(A + B)]$$

$$= \frac{1}{2}\sin 10°(\cos 20° - \cos 120°) = \frac{1}{2}\sin 10°\left(\cos 20° + \frac{1}{2}\right)$$

$$\left[\because \cos 120° = \cos(180° - 60°) = -\cos 60° = -\frac{1}{2}\right]$$

$$= \frac{1}{2}\sin 10° \cos 20° + \frac{1}{4}\sin 10° = \frac{1}{4}(2\sin 10° \cos 20°) + \frac{1}{4}\sin 10°$$

$$= \frac{1}{4}[\sin(10° + 20°) + \sin(10 - 20°)] + \frac{1}{4}\sin 10°$$

$$= \frac{1}{4}[\sin 30° + \sin(-10°)] + \frac{1}{4}\sin 10$$

$$= \frac{1}{4}\left(\frac{1}{2} - \sin 10°\right) + \frac{1}{4}\sin 10° \qquad [\because \sin(-\theta) = -\sin\theta]$$

$$= \frac{1}{8} - \frac{1}{4}\sin 10° + \frac{1}{4}\sin 10° = \frac{1}{8} = \text{R.H.S.}$$

Example 9. *Prove that:*

$$\tan 20° \tan 40° \tan 80° = \tan 60°$$

Solution: L.H.S. $= \tan 20° \tan 40° \tan 80° = \dfrac{\sin 20° \sin 40° \sin 80°}{\cos 20° \cos 40° \cos 80°}$

$$= \frac{(2\sin 20° \sin 40°)\sin 80°}{(2\cos 20° \cos 40°)\cos 80°} = \frac{(\cos 20° - \cos 60°)\sin 80°}{(\cos 60° + \cos 20°)\cos 80°}$$

$$= \frac{\sin 80° \cos 20° - (1/2)\sin 80°}{(1/2)\cos 80° + \cos 80° \cos 20°} = \frac{2\sin 80° \cos 20° - \sin 80°}{\cos 80° + 2\cos 80° \cos 20°}$$

$$= \frac{\sin 100° + \sin 60° - \sin 80°}{\cos 80° + \cos 100° + \cos 60°} = \frac{\sin(180° - 80°) + \sin 60° - \sin 80°}{\cos 80° + \cos(180° - 80°) + \cos 60°}$$

$$= \frac{\sin 80° + \sin 60° - \sin 80°}{\cos 80° - \cos 80° + \cos 60°} = \frac{\sin 60°}{\cos 60°} = \tan 60° = \text{RHS.}$$

Example 10. *Prove that:* $4 \sin A \sin (60° - A) \sin (60° + A) = \sin 3A$.

Hence deduce that: $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$

Solution: $4 \sin A \sin (60° - A) \sin (60° + A) = 2 \sin A[2 \sin (60° - A) \sin (60° + A)]$

$= 2 \sin A[\cos \{(60° - A) - (60° + A)\} - \cos \{(60° - A) + (60° + A)\}]$

$[\because 2 \sin A \sin B = \cos (A - B) - \cos (A + B)]$

$= 2 \sin A[\cos (-2A) - \cos 120°]$

$= 2 \sin A \cos 2A - 2 \sin A \cos 120°$ $\quad [\because \cos (-2A) = \cos 2A]$

$= [\sin (A + 2A) + \sin (A - 2A)] - 2 \sin A\left(-\frac{1}{2}\right)$

$\left[\because 2 \sin A \cos B = \sin (A + B) + \sin (A - B) \text{ and } \cos 120° = \frac{1}{2}\right]$

$= \sin 3A + \sin (-A) + \sin A$

$= \sin 3A - \sin A + \sin A$ $\quad [\because \sin (-A) = -\sin A]$

$= \sin 3A$

or $\quad 4 \sin A \sin (60° - A) \sin (60° + A) = \sin 3A$ ...(1)

Putting $A = 20°$, we get

$$4 \sin 20° \sin 40° \sin 80° = \sin 60° = \frac{\sqrt{3}}{2}$$

or $\quad \sin 20° \sin 40° \sin 80° = \frac{\sqrt{3}}{8}$ or $\sin 20° \sin 40° \frac{\sqrt{3}}{2} \sin 80° = \frac{\sqrt{3}}{8} \times \frac{\sqrt{3}}{2}$

[Multiplying both sides by $\frac{\sqrt{3}}{2}$]

or $\quad \sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$ $\quad \left[\because \frac{\sqrt{3}}{2} = \sin 60°\right]$

Example 11. *Prove that*

(i) $\frac{\sin 7A - \sin 5A}{\cos 7A + \cos 5A} = \tan A$; (ii) $\frac{\cos A - \cos 3A}{\sin 3A - \sin A} = \tan 2A$;

(iii) $\frac{\sin A + \sin 2A}{\cos A - \cos 2A} = \cot \frac{A}{2}$.

Solution: (i) L.H.S. $= \frac{\sin 7A - \sin 5A}{\cos 7A + \cos 5A} = \frac{2 \cos\left(\frac{7A + 5A}{2}\right).\sin\left(\frac{7A - 5A}{2}\right)}{2 \cos\left(\frac{7A + 5A}{2}\right).\cos\left(\frac{7A - 5A}{2}\right)}$

$= \frac{2 \cos 6A \sin A}{2 \cos 6A \cos A} = \frac{\sin A}{\cos A} = \tan A = \text{R.H.S.}$

(ii) L.H.S. $= \dfrac{\cos A - \cos 3A}{\sin 3A - \sin A} = \dfrac{2\sin\left(\dfrac{A+3A}{2}\right)\sin\left(\dfrac{3A-A}{2}\right)}{2\cos\left(\dfrac{3A+A}{2}\right)\sin\left(\dfrac{3A-A}{2}\right)}$

$$= \frac{\sin 2A \sin A}{\cos 2A \sin A} = \frac{\sin 2A}{\cos 2A} = \tan 2A = \text{R.H.S.}$$

(iii) L.H.S. $= \dfrac{\sin A + \sin 2A}{\cos A - \cos 2A} = \dfrac{2\sin\left(\dfrac{A+2A}{2}\right)\cos\left(\dfrac{2A-A}{2}\right)}{2\sin\left(\dfrac{A+2A}{2}\right)\sin\left(\dfrac{2A-A}{2}\right)}$

$$= \frac{\cos\frac{A}{2}}{\sin\frac{A}{2}} = \cot\frac{A}{2} = \text{R.H.S.}$$

Example 12. *Prove that*

(i) $\dfrac{\sin(\alpha+\beta) - 2\sin\alpha + \sin(\alpha-\beta)}{\cos(\alpha+\beta) - 2\cos\alpha + \cos(\alpha-\beta)} = \tan\alpha;$

(ii) $\dfrac{\sin A + \sin 2A + \sin 4A + \sin 5A}{\cos A + \cos 2A + \cos 4A + \cos 5A} = \tan 3A;$

(iii) $\dfrac{\cos\theta - \cos 2\theta + \cos 3\theta}{\sin\theta - \sin 2\theta + \sin 3\theta} = \cot 2\theta;$

(iv) $\dfrac{\sin 11\theta \sin\theta + \sin 7\theta \sin 3\theta}{\cos 11\theta \sin\theta + \cos 7 \sin 3\theta} = \tan 8\theta.$

Solution: (i) $\dfrac{\sin(\alpha+\beta) - 2\sin\alpha + \sin(\alpha-\beta)}{\cos(\alpha+\beta) - 2\cos\alpha + \cos(\alpha-\beta)}$

$$= \frac{[\sin(\alpha+\beta) + \sin(\alpha-\beta)] - 2\sin\alpha}{[\cos(\alpha+\beta) + \cos(\alpha-\beta)] - 2\cos\alpha} = \frac{2\sin\alpha\cos\beta - 2\sin\alpha}{2\cos\alpha\cos\beta - 2\cos\alpha}$$

$$= \frac{2\sin\alpha(\cos\beta - 1)}{2\cos\alpha(\cos\beta - 1)} = \frac{\sin\alpha}{\cos\alpha} = \tan\alpha$$

(ii) $\dfrac{\sin A + \sin 2A + \sin 4A + \sin 5A}{\cos A + \cos 2A + \cos 4A + \cos 5A} = \dfrac{(\sin 5A + \sin A) + (\sin 4A + \sin 2A)}{(\cos 5A + \cos A) + (\cos 4A + \cos 2A)}$

$$= \frac{2\sin 3A\cos 2A + 2\sin 3A\cos A}{2\cos 3A\cos 2A + 2\cos 3A\cos A} = \frac{2\sin 3A(\cos 2A + \cos A}{2\cos 3A(\cos 2A + \cos A}$$

$$= \frac{\sin 3A}{\cos 3A} = \tan 3A$$

(iii) $\dfrac{\cos\theta-\cos 2\theta+\cos 3\theta}{\sin\theta-\sin 2\theta+\sin 3\theta} = \dfrac{(\cos 3\theta+\cos\theta)-\cos 2\theta}{(\sin 3\theta+\sin\theta)-\sin 2\theta}$

$$= \frac{2\cos\frac{3\theta+\theta}{2}\cos\frac{3\theta-\theta}{2}-\cos 2\theta}{2\sin\frac{3\theta+\theta}{2}\cos\frac{3\theta-\theta}{2}-\sin 2\theta}$$

$$= \frac{2\cos 2\theta\cos\theta-\cos 2\theta}{2\sin 2\theta\cos\theta-\sin 2\theta} = \frac{\cos 2\theta(2\cos\theta-1)}{\sin 2\theta(2\cos\theta-1)}$$

$$= \frac{\cos 2\theta}{\sin 2\theta} = \cot 2\theta$$

(iv) $\dfrac{\sin 11\theta\sin\theta+\sin 7\theta\sin 3\theta}{\cos 11\theta\sin\theta+\cos 7\theta\sin 3\theta}$ (Multiplying the num. and denom. by 2)

$$= \frac{2\sin 11\theta\sin\theta+2\sin 7\theta\sin 3\theta}{2\cos 11\theta\sin\theta+2\cos 7\theta\sin 3\theta}$$

$$= \frac{(\cos 10\theta-\cos 12\theta)+(\cos 4\theta-\cos 10\theta)}{(\sin 12\theta-\sin 10\theta)+(\sin 10\theta-\sin 4\theta)} = \frac{\cos 4\theta-\cos 12\theta}{\sin 12\theta-\sin 4\theta}$$

$$= \frac{2\sin 8\theta\sin 4\theta}{2\cos 8\theta\sin 4\theta} = \frac{\sin 8\theta}{\cos 8\theta} = \tan 8\theta$$

Example 13. *Prove that* $\dfrac{\sin 38^\circ-\cos 68^\circ}{\cos 68^\circ+\sin 38^\circ} = \sqrt{3}\tan 8^\circ$

Solution: $\dfrac{\sin 38^\circ-\cos 68^\circ}{\cos 68^\circ+\sin 38^\circ} = \dfrac{\sin(90^\circ-52^\circ)-\cos 68^\circ}{\cos 68^\circ+\sin(90^\circ-52^\circ)} = \dfrac{\cos 52^\circ-\cos 68^\circ}{\cos 68^\circ+\cos 52^\circ}$

$$= \frac{2\sin\left(\frac{52^\circ+68^\circ}{2}\right)\sin\left(\frac{68^\circ-52^\circ}{2}\right)}{2\cos\left(\frac{68^\circ+52^\circ}{2}\right)\cos\left(\frac{68^\circ-52^\circ}{2}\right)} = \frac{\sin 60^\circ\sin 8^\circ}{\cos 60^\circ\cos 8^\circ}$$

$$= \tan 60^\circ\tan 8^\circ = \sqrt{3}\tan 8^\circ = \text{R.H.S.}$$

Example 14. *Prove that* $\cos\theta+\cos\left(\frac{2\pi}{3}-\theta\right)+\cos\left(\frac{2\pi}{3}+\theta\right) = 0$

Solution: $\cos\theta+\left[\cos\left(\frac{2\pi}{3}-\theta\right)+\cos\left(\frac{2\pi}{3}+\theta\right)\right]$

$$= \cos\theta+2\cos\frac{\frac{2\pi}{3}-\theta+\frac{2\pi}{3}+0}{2}\cos\frac{\frac{2\pi}{3}-\theta-\frac{2\pi}{3}-\theta}{2}$$

$$= \cos\theta+2\cos\frac{2\pi}{3}\cos(-\theta) \qquad \left[\because \cos\frac{2\pi}{3}=\cos 120^\circ=-\frac{1}{2};\cos(-\theta)=\cos\theta\right]$$

$$= \cos\theta+2\left(-\frac{1}{2}\right)\cos\theta = \cos\theta-\cos\theta = 0$$

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3. Prove that:

(i) $2\cos 2B\cos(A-B)=\cos(A+B)+\cos(A-3B)$;

(ii) $2\sin(2\theta+\phi)\cos(\theta-2\phi)=\sin(3\theta-\phi)+\sin(\theta+3\phi)$;

(iii) $\cos(60^\circ+\alpha)\sin(60^\circ-\alpha)=\frac{1}{4}(\sqrt{3}-2\sin 2\alpha)$;

(iv) $4\cos\theta\cos\left(\frac{\pi}{3}-\theta\right)\cos\left(\frac{\pi}{3}+\theta\right)=\cos 3\theta$ (v) $\sin(45^\circ+A)\sin(45^\circ-A)=\frac{1}{2}\cos 2A$;

(vi) $\sec\left(\frac{\pi}{4}+\theta\right)\sec\left(\frac{\pi}{4}-\theta\right)=2\sec 2\theta$;

(vii) $\sin 3x+\sin 2x-\sin x=4\sin x\cos\frac{x}{2}\cos\frac{3x}{2}$.

4. Prove that:

(i) $\cos(A+B)+\sin(A-B)=2\sin(45^\circ+A)\cos(45^\circ+B)$;

(ii) $\cos 15^\circ-\sin 15^\circ=\frac{1}{\sqrt{2}}$;

(iii) $\tan(A+30^\circ)+\cot(A-30^\circ)=\frac{1}{\sin 2A-\sin 60^\circ}$;

(iv) $\tan(45^\circ+\theta)+\tan(45^\circ-\theta)=2\sec 2\theta$;

(v) $\tan(45^\circ+\theta)-\tan(45^\circ-\theta)=2\tan 2\theta$;

(vi) $\cos\theta+\cos\left(\frac{2\pi}{3}-\theta\right)+\cos\left(\frac{2\pi}{3}+\theta\right)=0$;

(vii) $\sin\theta+\sin\left(\theta+\frac{2\pi}{3}\right)+\sin\left(\theta+\frac{4\pi}{3}\right)=0$;

(viii) $\cos\theta+\cos\left(\frac{2\pi}{3}+\theta\right)+\cos\left(\frac{4\pi}{3}+\theta\right)=0$;

(ix) $(\sin 3A+\sin A)\sin A+(\cos 3A-\cos A)\cos A=0$;

(x) $\cos(A+B)+\sin(A-B)=25\sin\left(\frac{\pi}{4}+A\right)\cos\left(\frac{\pi}{4}+B\right)$;

(xi) $\sin 2x+2\sin 4x+\sin 6x=4\cos^2 x\sin 4x$;

(xii) $\cot 4x(\sin 5x+\sin 3x)=\cot x(\sin 5x-\sin 3x)$.

5. Prove that:

(i) $\sin 10^\circ\sin 30^\circ\sin 50^\circ\sin 70^\circ=\frac{1}{16}$; (ii) $\cos 20^\circ\cos 40^\circ\cos 80^\circ=\frac{1}{8}$;

(iii) $\sin 20^\circ\sin 40^\circ\sin 80^\circ=\frac{\sqrt{3}}{8}$; (iv) $\cos 10^\circ\cos 50^\circ\cos 70^\circ=\frac{\sqrt{3}}{8}$;

(v) $\tan 20^\circ\tan 30^\circ\tan 40^\circ\tan 80^\circ=\frac{1}{\sqrt{3}}$.

6. Prove that:

(i) $\frac{\sin A+\sin B}{\cos A+\cos B}=\tan\left(\frac{A+B}{2}\right)$; (ii) $\frac{\cos 2B-\cos 2A}{\sin 2B+\sin 2A}=\tan(A-B)$;

(iii) $\frac{\cos A-\cos 3A}{\sin 3A-\sin A}=\tan 2A$; (iv) $\frac{\sin 7A+\sin 3A}{\cos 7A+\cos 3A}=\tan 5A$;

11. (i) Using $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$ on the two expressions on the left side, we get

$$\text{L.H.S.} = \frac{-1}{2}[\cos(B + C + B - C) - \cos(B + C - (B - C))]$$

$$- \frac{1}{2}[\cos(C + A + C - A) - \cos(C + A - (C - A))]$$

$$= \frac{-1}{2}[\cos 2B - \cos 2C + \cos 2C - \cos 2A] = \frac{-1}{2}[\cos 2B - \cos 2A]$$

$$= \frac{-1}{2}\left[2\sin\left(\frac{2B + 2A}{2}\right)\sin\left(\frac{2A - 2B}{2}\right)\right] = -\sin(A + B)\sin(A - B) = \text{R.H.S.}$$

(ii) Using $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$ on each expression on the left side, we get

$$\text{L.H.S.} = \frac{1}{2}[\cos(A - D - (B - C)) - \cos(A - D + B - C) + \cos(B - D - (C - A))$$

$$- \cos(B - D + C - A) + \cos(C - D - (A - B)) - \cos(C - D + A - B)]$$

$$= \frac{1}{2}[\cos(A + C - B - D) - \cos(A + B - C - D) + \cos(A + B - C - D)$$

$$- \cos(B + C - A - D) + \cos(B + C - A - D) - \cos(A + C - B - D)]$$

$$= \frac{1}{2}(0) = 0 = \text{R.H.S.}$$

14. $\because$ A, B, C are in A.P. $\therefore \frac{A + C}{2} = B$ ...(1)

Now, $$\frac{\sin A - \sin C}{\cos C - \cos A} = \frac{2\cos\frac{A + C}{2}\sin\frac{A - C}{2}}{2\sin\frac{A + C}{2}\sin\frac{A - C}{2}} = \frac{\cos\frac{A + C}{2}}{\sin\frac{A + C}{2}}$$

$$= \cot\left(\frac{A + C}{2}\right) = \cot B \qquad \left[\because \text{From (1) } \frac{A + C}{2} = B\right]$$

15. $$\text{L.H.S.} = \left\{\frac{\cos A + \cos B}{\sin A - \sin B}\right\}^n + \left\{\frac{\sin A + \sin B}{(\cos A - \cos B)}\right\}^n$$

$$= \left[\frac{2\cos\frac{A + B}{2}\cos\frac{A - B}{2}}{2\cos\frac{A + B}{2}\cdot\sin\frac{A - B}{2}}\right]^n + \left[\frac{2\sin\frac{A + B}{2}\cdot\cos\frac{A - B}{2}}{-2\sin\frac{A + B}{2}\cdot\sin\frac{A - B}{2}}\right]^n$$

$$= \left(\cot\frac{A - B}{2}\right)^n + \left(-\cot\frac{A - B}{2}\right)^n$$

When n is odd, $\left(-\cot\frac{A - B}{2}\right)^n = -\cot^n\frac{A - B}{2}$, therefore,

$$\text{L.H.S.} = \cot^n\frac{A - B}{2} - \cot^n\frac{A - B}{2} = 0 = \text{R.H.S.}$$

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Solution: (i) L.H.S. $= \sin A \sin (60° - A) \sin (60° + A)$

$$= \frac{1}{2}\sin A\,[2 \sin (60° - A) \sin (60° + A)]$$

$$= \frac{1}{2}\sin A\,[\cos(-2A) - \cos 120°] = \frac{1}{2}\sin A\left[\cos 2A - \left(-\frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\sin A\left(\cos 2A + \frac{1}{2}\right) = \frac{1}{4}\sin A(2\cos 2A + 1)$$

$$= \frac{1}{4}\sin A\,[2(1 - 2\sin^2 A) + 1] = \frac{1}{4}\sin A(3 - 4\sin^2 A)$$

$$= \frac{1}{4}(3\sin A - 4\sin^3 A) = \frac{1}{4}(\sin 3A) = \frac{1}{4}\sin 3A$$

(ii) L.H.S. $= \sin^2\theta + \sin^2(60° + \theta) + \sin^2(60° - \theta)$

$$= \frac{1-\cos 2\theta}{2} + \frac{1-\cos(120° + 2\theta)}{2} + \frac{1-\cos(120° - 2\theta)}{2}$$

$$= \frac{3}{2} - \frac{1}{2}[\cos 2\theta + \cos(120° + 2\theta) + \cos(120° - 2\theta)]$$

$$= \frac{3}{2} - \frac{1}{2}\left[\cos 2\theta + 2\cos\frac{120° + 2\theta + 120° - 2\theta}{2} \times \cos\frac{120° + 2\theta - 120° + 2\theta}{2}\right]$$

$$= \frac{3}{2} - \frac{1}{2}(\cos 2\theta + 2\cos 120° \cos 2\theta) \quad \left[\because \cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}\right]$$

$$= \frac{3}{2} - \frac{1}{2}\left[\cos 2\theta + 2\left(-\frac{1}{2}\right)\cos 2\theta\right]$$

$$\left[\because \cos 120° = \cos(180° - 60°) = -\cos 60° = -\frac{1}{2}\right]$$

$$= \frac{3}{2} - \frac{1}{2}(\cos 2\theta - \cos 2\theta) = \frac{3}{2} = \text{R.H.S.}$$

(iii) L.H.S. $= \cos^2\alpha + \cos^2(\alpha + 120°) + \cos^2(\alpha - 120°)$

$$= \frac{1+\cos 2\alpha}{2} + \frac{1+\cos(2\alpha + 240°)}{2} + \frac{1+\cos(2\alpha - 240°)}{2}$$

$$= \frac{3}{2} + \frac{1}{2}[\cos 2\alpha + \cos(2\alpha + 240°) + \cos(2\alpha - 240°)]$$

$$= \frac{3}{2} + \frac{1}{2}\left[\cos 2\alpha + 2\cos\frac{2\alpha + 240° + 2\alpha - 240°}{2} \times \cos\frac{2\alpha + 240° - 2\alpha + 240°}{2}\right]$$

$$= \frac{3}{2} + \frac{1}{2}(\cos 2\alpha + 2\cos 2\alpha \cos 240°)$$

$$= \frac{3}{2} + \frac{1}{2}\left[\cos 2\alpha + 2\cos 2\alpha\left(-\frac{1}{2}\right)\right]$$

$$\left[\because \cos 240° = \cos(180° + 60°) = \cos 60° = -\frac{1}{2}\right]$$

$$= \frac{3}{2} + \frac{1}{2}(\cos 2\alpha - \cos 2\alpha) = \frac{3}{2} = \text{R.H.S.}$$

(iii) $\tan x = -\dfrac{4}{3}, \dfrac{\pi}{2} < x < \pi \Rightarrow \dfrac{\pi}{4} < \dfrac{x}{2} < \dfrac{\pi}{2}$

i.e., $\dfrac{x}{2}$ lies in the first quadrant, so that all *t*-ratios of $\dfrac{x}{2}$ are positive.

Also $\quad \cos^2 x = \dfrac{1}{\sec^2 x} = \dfrac{1}{1+\tan^2 x} = \dfrac{1}{1+\dfrac{16}{9}} = \dfrac{9}{25}$

and $\cos x$ is negative in the second quadrant.

$$\Rightarrow \quad \cos x = -\frac{3}{5}$$

$$\therefore \quad \sin\frac{x}{2} = \sqrt{\frac{1-\cos x}{2}} = \sqrt{\frac{1-(-3/5)}{2}} = \sqrt{\frac{4}{5}} = \frac{2}{\sqrt{5}}$$

$$\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}} = \sqrt{\frac{1-3/5}{2}} = \frac{1}{\sqrt{5}}$$

$$\tan\frac{x}{2} = \frac{\sin x/2}{\cos x/2} = \frac{2/\sqrt{5}}{1/\sqrt{5}} = 2$$

Example 21. *If* $\cos\alpha = \dfrac{3}{5}$ and $\cos\beta = \dfrac{4}{5}$, *find the value of* $\cos\dfrac{\alpha-\beta}{2}$, α and β *being positive acute angles.*

Solution: We have

$$2\cos^2\frac{\alpha-\beta}{2} = 1 + \cos(\alpha-\beta) \qquad [\because \cos\theta = 2\cos^2\theta/2 - 1]$$

or
$$2\cos^2\frac{\alpha-\beta}{2} = 1 + \cos\alpha\cos\beta + \sin\alpha\sin\beta \qquad \text{...(i)}$$

Now, $\quad \cos\alpha = \dfrac{3}{5} \quad \therefore \quad \sin\alpha = \sqrt{1-\cos^2\alpha} = \sqrt{1-\dfrac{9}{25}} = \sqrt{\dfrac{16}{25}} = \dfrac{4}{5}$

and $\quad \cos\beta = \dfrac{4}{5} \quad \therefore \quad \sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-\dfrac{16}{25}} = \sqrt{\dfrac{9}{25}} = \dfrac{3}{5}$

Substituting these values in (1), we get

$$2\cos^2\frac{\alpha-\beta}{2} = 1 + \frac{3}{5}, \frac{4}{5} + \frac{4}{5}\cdot\frac{3}{5} = 1 + \frac{12}{25} + \frac{12}{25} = 1 + \frac{24}{25} = \frac{49}{25}$$

or
$$\cos^2\frac{\alpha-\beta}{2} = \frac{49}{50} \Rightarrow \cos\frac{\alpha-\beta}{2} = \frac{7}{\sqrt{50}} = \frac{7}{5\sqrt{2}}$$

Example 22. *If α and β be two distinct angles satisfying the equation* $a\cos\phi + b\sin\phi = c$, *prove that*

(i) $\cos(\alpha+\beta) = \dfrac{a^2-b^2}{a^2+b^2}$; $\qquad$ (ii) $\sin(\alpha+\beta) = \dfrac{2ab}{a^2+b^2}$;

(iii) $\tan(\alpha+\beta) = \dfrac{2ab}{a^2-b^2}$.

Solution: L.H.S. $= \cos 36° \cos 72° \cos 108° \cos 144°$

$$= \cos 36° \cos (90° - 18°) \cos (90° + 18°) \cos 144°$$

$$= \cos 36° \sin 18°(-\sin 18°)(-\cos 36°)$$

$$= \sin^2 18° \cos^2 36° = \left(\frac{\sqrt{5}-1}{4}\right)^2 \left(\frac{\sqrt{5}+1}{4}\right)^2$$

$$= \left(\frac{5-1}{16}\right)^2 = \left(\frac{1}{4}\right)^2 = \frac{1}{16} = \text{R.H.S.}$$

Example 27. *Prove that* $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right) = \frac{1}{8}$.

Solution: L.H.S. $= \left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left[1+\cos\left(\pi-\frac{3\pi}{8}\right)\right]\left[1+\cos\left(\pi-\frac{\pi}{8}\right)\right]$

$$= \left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right)$$

$[\because \cos(\pi-\theta) = -\cos\theta]$

$$= \left(1-\cos^2\frac{\pi}{8}\right)\left(1-\cos^2\frac{3\pi}{8}\right) = \left(1-\cos^2\frac{\pi}{8}\right)\left[1-\cos^2\left(\frac{\pi}{2}-\frac{\pi}{8}\right)\right]$$

$$= \left(1-\cos^2\frac{\pi}{8}\right)\left(1-\sin^2\frac{\pi}{8}\right) \qquad \left[\because \cos\left(\frac{\pi}{2}-\theta\right) = \sin\theta\right]$$

$$= \left(1-\frac{2+\sqrt{2}}{4}\right)\left(1-\frac{2-\sqrt{2}}{4}\right)$$

$$\left[\because \cos\frac{\pi}{8} = \sin 22\frac{1}{2}° = \frac{\sqrt{2+\sqrt{2}}}{2} \text{ and } \sin\frac{\pi}{8} = \sin 22\frac{1}{2}° = \frac{\sqrt{2+\sqrt{2}}}{2}\right]$$

$$= \left(\frac{2-\sqrt{2}}{4}\right)\left(\frac{2+\sqrt{2}}{4}\right) = \frac{(2)^2-(\sqrt{2})^2}{16} = \frac{4-2}{16} = \frac{2}{16} = \frac{1}{8}$$

Example 28. *Prove that*

(i) $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{16}$;

(ii) $\cos\frac{\pi}{15}\cos\frac{2\pi}{15}\cos\frac{3\pi}{15}\cos\frac{4\pi}{15}\cos\frac{5\pi}{15}\cos\frac{6\pi}{15}\cos\frac{7\pi}{15} = \frac{1}{128}$;

(iii) $\sin^4\frac{\pi}{8}+\sin^4\frac{3\pi}{8}+\sin^4\frac{5\pi}{8}+\sin^4\frac{7\pi}{8} = \frac{3}{2}$.

Solution: (i) Let

$$x = \cos\frac{\pi}{15}\cdot\cos\frac{2\pi}{15}\cdot\cos\frac{4\pi}{15}\cdot\cos\frac{7\pi}{15}.$$

Multiplying both sides by $\sin\frac{\pi}{15}$, we get

17. Prove that:

(i) $\sin 20° \sin 40° \sin 60° \sin 80° = \sin\frac{\pi}{9}\sin\frac{2\pi}{9}\sin\frac{3\pi}{9}\sin\frac{4\pi}{9} = \frac{3}{16}$;

(ii) $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$;

(iii) $\tan 20° \tan 40° \tan 60° \tan 80° = 3$.

18. Show that:

(i) $\tan 6° \tan 42° \tan 66° \tan 78° = 1$;. (ii) $\tan 9° \tan 27° \tan 63° \tan 81° = 1$.

19. Prove that: $\cos 6A = 32\cos^6 A - 48\cos^4 A + 18\cos^2 A - 1$

20. Find sine, cosine and tangent of $\frac{x}{2}$ if $0 \le x \le 2\pi$ in the following problems:

(i) $\sin x = -\frac{1}{2}$, x in Quadrant IV.

(ii) $\tan x = \frac{3}{4}$, x is in Quadrant III.

LEVEL OF DIFFICULTY B

21. If $\tan A = \frac{1-\cos B}{\sin B}$, find whether $\tan 2A = \tan B$ or not.

22. Prove that $2\sin\theta\cos^3\theta - 2\sin^3\theta\cos\theta = \frac{1}{2}\sin 4\theta$.

23. If $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$, find the value of $\tan(2A + B)$.

24. Prove that (i) $\sin A \sin(60° - A)\sin(60° + A) = \frac{1}{4}\sin 3A$;

(ii) $\cos A \cos(60° - A)\cos(60° + A) = \frac{1}{4}\cos 3A$;

25. If $2\tan\alpha = 3\tan\beta$, show that $\tan(\alpha - \beta) = \frac{\sin 2\beta}{5 - \cos 2\beta}$.

26. Prove that $\frac{1}{\sin 10°} - \frac{\sqrt{3}}{\cos 10°} = 4$.

27. If $\sec(\phi + \alpha) + \sec(\phi - \alpha) = 2\sec\phi$, show that $\cos\phi = \sqrt{2}\cos\frac{\alpha}{2}$.

28. Prove that $\sec\left(\frac{\pi}{4} + \theta\right)\sec\left(\frac{\pi}{4} - \theta\right) = 2\sec 2\theta$.

29. In a triangle ABC, if $\sin 2A = \sin 2B + \sin 2C$, prove that either $B = 90°$ or $C = 90°$.

30. Prove that:

(i) $\cos 6° \cos 42° \cos 66° \cos 78° = \frac{1}{16}$; (ii) $\sin 12° \sin 48° \sin 54° = \frac{1}{8}$.

31. Prove that:

(i) $\sin^2 72° - \sin^2 60° = \frac{1}{8}(\sqrt{5} - 1)$; (ii) $\cos^2 48° - \sin^2 12° = \frac{1}{8}(\sqrt{5} + 1)$.

32. Prove that: $\cos 12° + \cos 60° + \cos 84° = \cos 24° + \cos 48°$

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When $\sin\frac{\theta-\alpha}{2}=0$,

$$\frac{6-\alpha}{2}=n\pi, \quad \text{or} \quad \theta=2n\pi+\alpha \qquad ...(2)$$

From (1) and (2), we have

$$\theta=2\,n\pi\pm\alpha, \text{ where } n=0, \pm1, \pm2, ...$$

General solution of equation $\tan\theta=\tan\alpha$:

Given $\quad \tan\theta=\tan\alpha \quad$ or $\quad \frac{\sin\theta}{\cos\theta}=\frac{\sin\alpha}{\cos\alpha}$

or, $\quad \sin\theta\cos\alpha-\cos\theta\sin\alpha=0 \quad$ or $\quad \sin(\theta-\alpha)=0$

$\therefore \quad \theta-\alpha=n\pi \quad \therefore \quad \theta=n\pi+\alpha$, where $n=0, \pm1, \pm2, ...$

General solution of equations

(i) $\sin^2\theta=\sin^2\alpha$; (ii) $\cos^2\theta=\cos^2\alpha$; (iii) $\tan^2\theta=\tan^2\alpha$;

(i) Here $\sin^2\theta=\sin^2\alpha \Rightarrow \frac{1-\cos 2\theta}{2}=\frac{1-\cos 2\alpha}{2}$

$\Rightarrow 1-\cos 2\theta=1-\cos 2\alpha \Rightarrow \cos 2\theta=\cos 2\alpha$

$\therefore \quad 2\theta=2n\pi\pm 2\alpha$

Hence $\theta=n\pi\pm\alpha$, where n is any integer.

(ii) Here $\cos^2\theta=\cos^2\alpha \Rightarrow \frac{1+\cos 2\theta}{2}=\frac{1+\cos 2\alpha}{2}$

$\Rightarrow 1+\cos 2\theta=1+\cos 2\alpha \Rightarrow \cos 2\theta=\cos 2\alpha$

$\therefore \quad 2\theta=2n\pi\pm 2\alpha$

Hence $\theta=n\pi\pm\alpha$, where n is any integer.

(iii) Here $\tan^2\theta=\tan^2\alpha \Rightarrow \tan\theta=\pm\tan\alpha \Rightarrow \tan\theta=\tan(\pm\alpha)$

Hence $\theta=n\pi\pm\alpha$, where n is any integer

Thus,

$$\left.\begin{array}{l}\sin^2\theta=\sin^2\alpha\\ \cos^2\theta=\cos^2\alpha\\ \tan^2\theta=\tan^2\alpha\end{array}\right\} \Rightarrow \theta=n\pi\pm\alpha, \text{ where } n\in Z.$$

Solution of an equation of form $a\cos\theta+b\sin\theta=c$

The given equation is $a\cos\theta+b\sin\theta=c$. ...(1)

Divide throughout by $\sqrt{a^2+b^2}$, i.e., by $\sqrt{(\text{coeff. of }\cos\theta)^2+(\text{coeff. of }\sin\theta)^2}$, we get

$$\frac{a}{\sqrt{a^2+b^2}}\cos\theta+\frac{b}{\sqrt{a^2+b^2}}\sin\theta=\frac{c}{\sqrt{a^2+b^2}} \qquad ...(2)$$

Let α be the least +ve angle such that

$$\frac{a}{\sqrt{a^2+b^2}}=\cos\alpha \text{ and } \frac{b}{\sqrt{a^2+b^2}}=\sin\alpha$$

Therefore, Eq. (2) becomes

$$\cos\alpha\cos\theta+\sin\alpha\sin\theta=\frac{c}{\sqrt{a^2+b^2}}$$

$\Rightarrow \quad \cos(\theta-\alpha)=\cos\beta$ (say), where $\cos\beta=\frac{c}{\sqrt{a^2+b^2}}$.

$\therefore \quad \theta-\alpha=2n\pi\pm\beta \Rightarrow \theta=2n\pi\pm\beta+\alpha$, where $n\in Z$.

(vii) $4\cos\theta - 3\sec\theta = \tan\theta \Rightarrow 4\cos\theta - \dfrac{3}{\cos\theta} = \dfrac{\sin\theta}{\cos\theta}$

$$\Rightarrow 4\cos^2\theta - 3 = \sin\theta \Rightarrow 4(1-\sin^2\theta) - 3 = \sin\theta$$

$$\Rightarrow 4\sin^2\theta + \sin\theta - 1 = 0 \Rightarrow \sin\theta = \frac{-1\pm\sqrt{17}}{8}$$

Now, $\left|\dfrac{-1\pm\sqrt{17}}{8}\right| < 1$

∴ There exist angles α and β such that

$$0 < \alpha < \frac{\pi}{2}, -\frac{\pi}{2} < \beta < 0$$

and $\sin\alpha = \dfrac{-1+\sqrt{17}}{8}, \sin\beta = \dfrac{-1-\sqrt{17}}{8}$

Now, $\sin\theta = \dfrac{-1+\sqrt{17}}{8} = \sin\alpha$

$\Rightarrow$ $\theta = n\pi + (-1)^n\alpha = n\pi + (-1)^n \sin^{-1}\left(\dfrac{-1+\sqrt{17}}{8}\right)$

Also $\sin\theta = \dfrac{-1-\sqrt{17}}{8} = \sin\beta$

$\Rightarrow$ $\theta = n\pi + (-1)^n\beta = n\pi + (-1)^n \sin^{-1}\left(\dfrac{-1-\sqrt{17}}{8}\right)$

Hence $\theta = n\pi + (-1)^n\sin^{-1}\left(\dfrac{-1+\sqrt{17}}{8}\right)$

or $n\pi + (-1)^n\sin^{-1}\left(\dfrac{-1-\sqrt{17}}{8}\right)$, where $n \in Z$

Example 7. *Solve the following equations:*

(i) $\tan 2\theta \tan\theta = 1$; (ii) $\tan x + \tan 2x + \tan x \tan 2x = 1$;

(iii) $\tan\theta + \tan 4\theta + \tan 7\theta = \tan\theta \tan 4\theta \tan 7\theta$;

(iv) $\tan x + \tan 2x + \tan 3x = \tan x \tan 2x \tan 3x$;

(v) $\tan\theta + \tan 2\theta + \sqrt{3}\tan\theta\tan 2\theta = \sqrt{3}$.

Solution: (i) $\tan 2\theta \tan\theta = 1 \Rightarrow \dfrac{\sin 2\theta}{\cos 2\theta}.\dfrac{\sin\theta}{\cos\theta} = 1 \Rightarrow \sin 2\theta \sin\theta = \cos 2\theta \cos\theta$

$$\Rightarrow \cos 2\theta\cos\theta - \sin 2\theta\sin\theta = 0 \Rightarrow \cos(2\theta+\theta) = 0$$

$$\Rightarrow \cos 3\theta = 0 = \cos\frac{\pi}{2} \Rightarrow 3\theta = n\pi + \frac{\pi}{2} \Rightarrow \theta = \frac{n\pi}{3} + \frac{\pi}{6}, n \in Z.$$

(v) The given equation is

$$\tan\left(\frac{\pi}{4}+\theta\right) + \tan\left(\frac{\pi}{4}-\theta\right) = 4$$

$\Rightarrow \quad \dfrac{1+\tan\theta}{1-\tan\theta} + \dfrac{1-\tan\theta}{1+\tan\theta} = 4 \Rightarrow \dfrac{(1+\tan\theta)^2 + (1-\tan\theta)^2}{(1-\tan\theta)(1+\tan\theta)} = 4$

$\Rightarrow \quad \dfrac{2+2\tan^2\theta}{1-\tan^2\theta} = 4 \Rightarrow 2 + 2\tan^2\theta = 4 - 4\tan^2\theta$

$\Rightarrow \quad 6\tan^2\theta = 2 \Rightarrow \tan^2\theta = \dfrac{1}{3} = \left(\dfrac{1}{\sqrt{3}}\right)^2 = \tan^2\dfrac{\pi}{6}$

$\Rightarrow \quad \theta = n\pi \pm \pi/6$, where n is any integer.

(vi) $\sin 2x = 3\tan x \cos 2x \Rightarrow \dfrac{\sin 2x}{\cos 2x} = 3\tan x \Rightarrow \tan 2x = 3\tan x$

$\therefore \quad \dfrac{2\tan x}{1-\tan^2 x} = 3\tan x \Rightarrow 2\tan x = 3\tan x(1 - \tan^2 x)$

$$\Rightarrow \tan x\,(2 - 3 + 3\tan^2 x) = 0 \Rightarrow \tan x\,(3\tan^2 x - 1) = 0$$

Either $\tan x = 0$ or $3\tan^2 x - 1 = 0$

When $\tan x = 0 \Rightarrow x = n\pi$

When $3\tan^2 x - 1 = 0 \Rightarrow \tan^2 x = \dfrac{1}{3} \Rightarrow \tan^2 x = \left(\dfrac{1}{\sqrt{3}}\right)^2 = \tan^2\dfrac{\pi}{6}$

$\therefore \quad x = n\pi \pm \dfrac{\pi}{6}$.

Hence $x = n\pi$ or $n\pi \pm \dfrac{\pi}{6}$, where n is any integer +ve or –ve.

(vii) $4\sin x \sin 2x \sin 4x = \sin 3x$

$\Rightarrow 4\sin x \sin(3x - x)\sin(3x + x) = \sin 3x$

$\Rightarrow 4\sin x(\sin^2 3x - \sin^2 x) = 3\sin x - 4\sin^3 x$

$\Rightarrow 4\sin x \sin^2 3x - 4\sin^3 x = 3\sin x - 4\sin^3 x \Rightarrow 4\sin x\sin^2 3x = 3\sin x$

$\Rightarrow \sin x(4\sin^2 3x - 3) = 0 \Rightarrow \sin x = 0 \quad$ or $\quad 4\sin^2 3x - 3 = 0$

$\Rightarrow \sin x = 0 \quad$ or $\quad \sin^2 3x = \dfrac{3}{4}$

Now, $\sin x = 0 \Rightarrow x = n\pi,\ n \in Z$

And, $\sin^2 3x = \dfrac{3}{4} \Rightarrow \sin^2 3x = \left(\dfrac{\sqrt{3}}{2}\right)^2 \Rightarrow \sin^2 3x = \sin^2\dfrac{\pi}{3}$

$$\Rightarrow 3x = m\pi \pm \frac{\pi}{3},\ m \in Z \Rightarrow x = \frac{m\pi}{3} \pm \frac{\pi}{9}$$

Hence $\quad x = n\pi$ or $x = \dfrac{m\pi}{3} \pm \dfrac{\pi}{9}$, where $m, n \in Z$

EXERCISE 3.7

LEVEL OF DIFFICULTY A

1. Find the general solution of the following equations:
   (i) $\sin 7\theta = 0$; (ii) $\cos 6\theta = 0$; (iii) $\tan 5\theta = 0$;
   (iv) $\sin \theta = \frac{1}{2}$; (v) $\cos 2\theta = \frac{1}{\sqrt{2}}$; (vi) $\cos 5\theta = -\frac{1}{\sqrt{2}}$;
   (vii) $\tan 3\theta = -1$; (viii) $\sec 2\theta = -2$; (ix) $\operatorname{cosec} 4\theta = -\frac{2}{\sqrt{3}}$;
   (x) $\operatorname{cosec} 2\theta = 2$; (xi) $\cot 2\theta = \sqrt{3}$.
2. Find the general solution of the following equations:
   (i) $\sin 4\theta = \sin \theta$; (ii) $\cos m\theta = \cos n\theta$; (iii) $\cos 3\theta = \sin 2\theta$;
   (iv) $\tan 3\theta = \cot \theta$; (v) $\tan 2\theta = \cot 5\theta$; (vi) $\cos m\theta = \sin n\theta$;
   (vii) $\cos mx + \cos nx = 0$; (viii) $\tan 2\theta = \tan \frac{6}{2}$; (ix) $\tan m\theta + \cot n\theta = 0$;
   (x) $\sin 3\theta + \cos 2\theta = 0$. (xi) $\cos 4x = \cos 2x$; (xii) $\sin 2x + \cos x = 0$
3. Find the general solution of each of the following equations :
   (i) $\cos \theta + \cos 7\theta = \cos 4\theta$; (ii) $\sin \theta + \sin 3\theta = \sin 7\theta$;
   (iii) $\cos \theta - \cos 2\theta = \sin 3\theta$; (iv) $\sin 4x - \sin 2x = \cos 3x$;
   (v) $\sin x + \sin 3x + \sin 5x = 0$; (vi) $\sin 2x - 2\sin 3x + \sin 4x = 0$;
   (vii) $\cos \theta + \cos 3\theta + \cos 5\theta + \cos 7\theta = 0$; (viii) $\cos 3x + \cos x - \cos 2x = 0$.
4. Find the general solution of each of the following equations:
   (i) $\sin \theta + \cos \theta = \sqrt{2}$; (ii) $\sqrt{3} \cos \theta + \sin \theta = \sqrt{2}$;
   (iii) $\tan \theta + \sec \theta = \sqrt{3}$; (iv) $\operatorname{cosec} \theta - \cot \theta = \sqrt{3}$;
   (v) $\sqrt{3} \cos \theta + \sin \theta = \sqrt{2}$; (vi) $\sqrt{2} \sec \theta + \tan \theta = 1$.
5. Find the general solution of each of the following equations:
   (i) $3\tan^2\theta = 1$; (ii) $4\sin^2\theta = 1$; (iii) $\sec^2\theta = 4/3$;
   (iv) $\tan^2 2\theta = \cot^2\alpha$; (v) $\tan^2\theta + \cot^2\theta = 2$; (vi) $4\cot^2\theta = 3\operatorname{cosec}^2\theta$;
   (vii) $3(\sec^2\theta + \tan^2\theta) = 5$; (viii) $7\sin^2 x + 3\cos^2 x = 4$;
   (ix) $\operatorname{cosec}^2 2\theta = 4 \operatorname{cosec} 2\theta$; (x) $\sec^2\theta + \operatorname{cosec}^2\theta = 4$.
6. Find the general solution of each of the following equations:
   (i) $2\sin^2 x + \sqrt{3}\cos x + 1 = 0$; (ii) $4\cos^2\theta - 4\sin\theta = 1$;
   (iii) $4\sin^2\theta - 8\cos\theta + 1 = 0$; (iv) $\sec^2 2\theta = 1 - \tan 2\theta$;
   (v) $3\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta - 3\sin^2\theta = 0$; (vi) $2(\cos\theta + \sec\theta) = 5$;
   (vii) $2\cos^2 x + 3\sin x = 0$.
7. Solve the simultaneous equations for general values of x and y:
   (i) $\cos(x + 2y) = \frac{1}{2}$, $\cos(2x + y) = \frac{\sqrt{3}}{2}$;
   (ii) $\tan(x - y) = 1$, $\sec(x + y) = \frac{2}{\sqrt{3}}$.
8. Find the general solution of the equation $\sec\theta - \operatorname{cosec}\theta = \frac{4}{3}$.
9. Find the general solution of the equation $4\sin^4 x + \cos^4 x = 1$.

6. (i) $x = 2n\pi + \frac{5\pi}{6}$, $n \in Z$; (ii) $\theta = n\pi + (-1)^n \frac{\pi}{6}$, $n \in Z$;

(iii) $\theta = 2n\pi \pm \frac{\pi}{3}$, $n \in Z$; (iv) $\theta = \frac{n\pi}{2}$, $n \in Z$ or $\frac{m\pi}{2} - \frac{\pi}{8}$, $m \in Z$;

(v) $\theta = n\pi - \frac{\pi}{3}$ or $\theta = m\pi + \frac{\pi}{6}$, $m, n \in Z$;

(vi) $\theta = 2n\pi \pm \frac{\pi}{3}$, $n \in Z$.

(vii) $x = n\pi + (-1)^n \frac{7\pi}{6}, n \in Z$.

7. (i) $x = (2m - n)\frac{2\pi}{3} \pm \frac{\pi}{9} \mp \frac{\pi}{9}$, $y = (2n - m)\ \frac{2\pi}{3} \pm \frac{2\pi}{9} \mp \frac{\pi}{18}$, $m, n \in Z$;

(ii) $x = (2m + n)\frac{\pi}{2} \pm \frac{\pi}{12} + \frac{\pi}{8}$, $y = (2m - n)\ \frac{\pi}{2} \pm \frac{\pi}{12} - \frac{\pi}{8}$, $m, n \in Z$.

8. $\frac{n\pi}{2} + \frac{(-1)^n}{2} \sin^{-1} \frac{3}{4}$; $n \in Z$.

9. $x = 2n\pi \pm \cos^{-1}\sqrt{\frac{3}{5}}$.

10. Null set.

11. $x = \frac{5\pi}{12}$, $y = \frac{\pi}{6}$.

12. $\theta = m\pi$, $n\pi \pm \frac{\pi}{3}$, where m and n are integers.

14. $-\frac{3}{2} \le \alpha \le \frac{1}{2}$.

15. $\frac{n\pi}{2} + \frac{\pi}{8}$, $n \in Z$.

16. 45°, 60°, 75°.

17. (i) $x = 0$; (ii) no solution.

23. $x = n\pi$, $4n\pi - \frac{\pi}{2}$, $\frac{4n\pi}{3} + \frac{\pi}{6}$, $n \in Z$.

24. $x = (2n + 1)\ \frac{\pi}{6} - \frac{a+b+c}{3}$, $n \in Z$.

25. $x = \frac{n\pi}{4}, \frac{n\pi}{2} + \frac{\pi}{8}$, $n \in Z$.

26. $\phi = (2n + 1)\frac{\pi}{8}$, $n \in Z$.

27. (i) $\phi = 4n\pi \pm \frac{2\pi}{3}$; (ii) $\phi = (2n + 1)\pi$, $2n\pi + \frac{\pi}{6}$; (iii) $\theta = m \pm \frac{1}{6}$, $m \in Z$.

28. $\theta = n\pi \pm \frac{\pi}{3}$, $\phi = n\,\pi \pm \frac{\pi}{6}$, $n \in Z$.

29. $\theta = \frac{\pi}{6}, \frac{\pi}{3}, \frac{2\pi}{3}$ and $\frac{5\pi}{6}$.

30. $x = \pm 1$, $y = (4n - 1)\frac{\pi}{2}$.

31. $x = n\pi$, $\frac{n\pi}{2} + \frac{\alpha}{2} + \frac{1}{2}\ (-1)^n \beta$, where $\sin \beta = k \sin \alpha$ and $n \in Z$.

33. $x = \frac{\pi}{6}$, $y = \frac{5\pi}{6}$ or $x = \frac{5\pi}{6}$, $y = \frac{\pi}{6}$.

34. $\lambda \ge -3$, $x = 2n\,\pi \pm \alpha$ where $\cos \alpha = (\lambda + 3)^{1/2}$.

36. $x = \pm\ \frac{\pi}{3}, \pm \frac{2\pi}{3}$.

37. $\frac{\sqrt{3}-1}{2}$.

38. $\frac{\pi}{6}, \frac{\pi}{3}$.

39. $\frac{\pi}{6}$

40. $x = (2n + 1)\frac{\pi}{2}$ or $x = n\,\pi - \frac{\pi}{4}$.

41. $13\frac{\pi}{4}$.

42. $\theta = \frac{n\pi}{3} + \frac{\pi}{12}, n \in Z$.

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Replacing k by $k + 1$, we get $(k + 1) \cdot 3^{k+1}$ Adding it to both sides,

L.H.S. $= 1 \cdot 3 + 2 \cdot 3^2 + k \cdot 3^3 + \cdots + k \cdot 3^k + (k + 1) \cdot 3^{k+1}$

$$\text{R.H.S.} = \frac{(2k-1)\cdot 3^{k+1}+3}{4} + (k+1)\cdot 3^{k+1}$$

$$= \frac{(2k-1)\cdot 3^{k+1}+3+4(k+1)\cdot 3^{k+1}}{4}$$

$$= \frac{3^{k+1}[2k-1+4\,(k+1)]+3}{4}$$

$$= \frac{3^{k+1}(6k+3)+3}{4} = \frac{(2k+1)\cdot 3^{k+2}+3}{4}$$

$$= \frac{[2(k+1)-1]\cdot 3^{(k+1)+1}+3}{4}$$

Therefore, $P(n)$ is true for $n = k + 1$, i.e., $P(k + 1)$ is true whenever $P(k)$ is true. Hence, by principle of mathematical induction, $P(n)$ is true for all values of $n \in N$.

Example 20. *Prove by using principle of mathematical induction:*

$$1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = (n-1)2^{n+1} + 2$$

Solution: Let $P(n)$ be the given statement, i.e., $P(n)$: $1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + n \cdot 2^n = (n - 1)2^{n+1} + 2$

Putting $n = 1$, L.H.S. $= 1{\cdot}2 = 2$ and R.H.S $= 0 + 2 = 2$

Therefore, $P(n)$ is true for $n = 1$. Assume that $P(n)$ is true for $n = k$, i.e., $P(k)$ is true, i.e., $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + k \cdot 2^k = (k - 1) \cdot 2^{k+1} + 2$

Last term $= k \cdot 2^k$.

Replacing k by $k + 1$, we get the next term $= (k + 1) \cdot 2^{k+1}$. Adding it to both sides,

L.H.S. $= 1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + \cdots + k \cdot 2^{k+1} + (k + 1) \cdot 2^{k+1}$

R.H.S. $= (k - 1)\, 2^{k+1} + 2 + (k + 1) \cdot 2^{k+1}$

$= 2^{k+1} \cdot [k - 1 + k + 1] + 2 = 2k \cdot 2^{k+1} + 2$

$= k \cdot 2^{k+2} + 2$

Therefore, $P(n)$ is true for $n = k + 1$. Thus, $P(k + 1)$ is true whenever $P(k)$ is true. Hence, by principle of mathematical induction, $P(k)$ is true for all $n \in N$.

Example 21. *Prove by using principle of mathematical induction:*

$$\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\cdots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

Solution: Let $P(n)$ be the given statement, i.e.,

$$P(n):\ \frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+\cdots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

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(ii) $\left[i^{17}-\left(\frac{1}{i}\right)^{34}\right]^2 = \left[(i^2)^8\cdot i-\frac{1}{(i^2)^{17}}\right]^2 = \left[(-1)^8\cdot i-\frac{1}{(-1)^{17}}\right]^2$

$= (i+1)^2 = i^2+2i+1 = -1+2i+1 = 2i$

Example 4. *For a positive integer n, show that the expression* $(1-i)^n\left(1-\frac{1}{i}\right)^n$ *equals* 2^n.

Solution: We have

$$(1-i)^n\left(1-\frac{1}{i}\right)^n = (1-i)^n\left(\frac{i-1}{i}\right)^n = [(1-i)^2]^n\left(-\frac{1}{i}\right)^n$$

$$= (1+i^2-2i)^n\cdot\left(\frac{i^2}{i}\right)^n = (1-1-2i)^n\cdot(i)^n$$

$$= (-2i)^n\cdot(i)^n = (-2i^2)^n = (-2\times -1)^n = 2^n$$

Example 5. *Find the least positive integer n which will reduce* $\left(\frac{1+i}{1-i}\right)^n$ *to unity.*

Solution: Given expression $= \left[\frac{(1+i)^2}{(1-i)(1+i)}\right]^n = \left(\frac{1+i^2+2i}{1-i^2}\right)^n$

$$= \left[\frac{1-1+2i}{1-(-1)}\right]^n = \left(\frac{2i}{2}\right)^n = i^n$$

which is equal to 1 for least positive integer $n = 4$.

Example 6. *Show that* $i^n + i^{n+1} + i^{n+2} + i^{n+3} = 0,\ \forall\, n \in N.$

Solution: We have

$$i^n + i^{n+1} + i^{n+2} + i^{n+3} = i^n(1+i+i^2+i^3) = i^n(1+i-1-i) = i^n\cdot 0 = 0.$$

CAUTION

For any two real numbers a and b, $\sqrt{a}\times\sqrt{b} = \sqrt{ab}$ is true only when at least one of a and b is either zero or positive.

If both a and b are positive real numbers, then the calculation

$$\sqrt{-a}\times\sqrt{-b} = \sqrt{(-a)(-b)} = \sqrt{ab}$$

is wrong.

The correct calculation is:

$$\sqrt{-a}\times\sqrt{-b} = (\sqrt{-1}\sqrt{a})\ (\sqrt{-1}\sqrt{b}) = (i\sqrt{a})(i\sqrt{b})$$

$$= i^2(\sqrt{a}\times\sqrt{b}) = (-1)(\sqrt{ab}) = -\sqrt{ab}$$

Thus, the calculation $\sqrt{-2}\times\sqrt{-3} = \sqrt{(-2)\times(-3)} = \sqrt{6}$ is wrong.

The correct result is $\sqrt{-2}\times\sqrt{-3} = (i\sqrt{2})(i\sqrt{3}) = i^2(\sqrt{2}\times\sqrt{3}) = -\sqrt{6}$.

Example 7. *Evaluate the following:*

(i) $\sqrt{-25}\sqrt{36}$; (ii) $\sqrt{-16}\sqrt{-25}$.

Solution: (i) $\sqrt{-25}\cdot\sqrt{36} = (\sqrt{-1}\sqrt{25})\cdot\sqrt{36} = (i\cdot 5)\cdot 6 = 30i$

(ii) $\sqrt{-16}\cdot\sqrt{-25} = (\sqrt{-1}\sqrt{16})(\sqrt{-1}\sqrt{25}) = (i\cdot 4)(i\cdot 5) = 20i^2 = 20(-1) = -20$

But we know that addition of real numbers is commutative, i.e., $a + c = c + a$ and $b + d = d + b$. Thus,

$$z_1 + z_2 = z_2 + z_1.$$

Associative law *Associative law for addition holds for the set of complex numbers, i.e., for any three complex numbers z_1, z_2, z_3, we have $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.*

Proof. Let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$ and $z_3 = a_3 + b_3 i$ be three complex numbers, then

$$\begin{aligned}(z_1 + z_2) + z_3 &= [(a_1 + b_1 i) + (a_2 + b_2 i)] + (a_3 + b_3 i)\\ &= [(a_1 + a_2) + (b_1 + b_2)i] + (a_3 + b_3 i)\\ &= [(a_1 + a_2) + a_3] + [(b_1 + b_2) + b_3]i\\ &= [a_1 + (a_2 + a_3)] + [b_1 + (b_2 + b_3)]i\end{aligned}$$

[addition of real numbers is associative.]

$$\begin{aligned}&= (a_1 + b_1 i) + [(a_2 + a_3) + (b_2 + b_3)i]\\ &= (a_1 + b_1 i) + [(a_2 + b_2 i) + (a_3 + b_3 i)] = z_1 + (z_2 + z_3).\end{aligned}$$

Additive identity There exists a complex number $0 + i0$ such that for every complex number $x + iy$, $(x + iy) + (0 + i0) = (x + 0) + i(y + 0) = x + iy$. Here $0 + i0$ is called the *additive identity*.

Additive inverse For every complex number $x + iy$, there exists a complex number $-x - iy$ such that

$$(x + iy) + (-x - iy) = (x - x) + i(y - y) = 0 + i0 = \text{additive identity}$$

$-x - iy$ is called the *additive inverse* of $x + iy$.

Subtraction of Complex Numbers

For two complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, the subtraction of z_2 from z_1 is defined as:

$$z_1 - z_2 = z_1 + (-z_2) = (a_1 + ib_1) + (-a_2 - ib_2) = (a_1 - a_2) + i(b_1 - b_2)$$

Illustration 4. If $z_1 = 1 - i$ and $z_2 = 5 + 2i$, then

$$\begin{aligned}z_1 - z_2 &= (1 - i) - (5 + 2i) = (1 - i) + (-5 - 2i)\\ &= (1 - 5) + i(-1 - 2) = -4 - 3i\end{aligned}$$

Multiplication of Complex Numbers

Multiplication of two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ is defined as

$$z_1 z_2 = (a + ib)(c + id) = (ac - bd) + i(ad + bc)$$

i.e., $z_1 z_2 = [\text{Re}(z_1)\,\text{Re}(z_2) - \text{Im}(z_1)\,\text{Im}(z_2)] + i\,[\text{Re}(z_1)\,\text{Im}(z_2) + \text{Re}(z_2)\,\text{Im}(z_1)]$

Illustration 5. If $z_1 = 4 + 3i$ and $z_2 = 3 - 2i$, then

$$\begin{aligned}z_1 z_2 = (4 + 3i)(3 - 2i) &= [4 \times 3 - 3 \times (-2)] + i[4 \times (-2) + 3 \times 3]\\ &= 18 + i\end{aligned}$$

Note: The product of complex numbers can be easily computed if we actually carry out the multiplication as given follows:

$$\begin{aligned}(a + ib)(c + id) &= ac + iad + ibc + i^2 bd\\ &= ac + i(ad + bc) - bd \qquad (\because i^2 = -1)\\ &= (ac - bd) + i(ad + bc)\end{aligned}$$

(vii) $z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$

$\Rightarrow \overline{z_1 + z_2} = \overline{(a_1 + a_2) + i(b_1 + b_2)} = (a_1 + a_2) - i(b_1 + b_2)$

$= (a_1 - ib_1) + (a_2 - ib_2) = \overline{z_1} + \overline{z_2}$

Thus, $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(viii) $z_1 z_2 = (a_1 + ib_1)(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$

$\Rightarrow \overline{z_1 z_2} = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + a_2 b_1)$

Also, $\overline{z_1}\,\overline{z_2} = (a_1 - ib_1)(a_2 - ib_2) = (a_1 a_2 - b_1 b_2) - i(a_1 b_2 + a_2 b_1)$

Thus, $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

(ix) $\dfrac{z_1}{z_2} = \dfrac{a_1 + ib_1}{a_2 + ib_2} = \dfrac{(a_1 + ib_1)(a_2 - ib_2)}{(a_2 + ib_2)(a_2 - ib_2)} = \dfrac{(a_1 a_2 + b_1 b_2) + i(-a_1 b_2 + a_2 b_1)}{a_2^2 + b_2^2}$

$= \dfrac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\dfrac{(-a_1 b_2 + a_2 b_1)}{a_2^2 + b_2^2}$

$\Rightarrow \overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} - i\dfrac{-a_1 b_2 + a_2 b_1}{a_2^2 + b_2^2} = \dfrac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\dfrac{a_1 b_2 - a_2 b_1}{a_2^2 + b_2^2}$

Also $\dfrac{\overline{z_1}}{\overline{z_2}} = \dfrac{a_1 - ib_1}{a_2 - ib_2} = \dfrac{(a_1 - ib_1)(a_2 + ib_2)}{(a_2 - ib_2)(a_2 + ib_2)} = \dfrac{a_1 a_2 + b_1 b_2 + i(a_1 b_2 - a_2 b_1)}{a_2^2 + b_2^2}$

$= \dfrac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\dfrac{a_1 b_2 - a_2 b_1}{a_2^2 + b_2^2}$

Thus, $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\bar{z}_1}{\bar{z}_2}$

Trick(s) for Problem Solving

To write complex number $\dfrac{a + ib}{c + id}$ in the form $A + iB$.

We have, $\dfrac{a + ib}{c + id} = \dfrac{(a + ib)(c - id)}{(c + id)(c - id)}$

[Multiplying the Nr. and the Dr. by the conjugate of the Dr.]

$= \dfrac{(ac + bd) + i(bc - ad)}{c^2 + d^2} = \dfrac{ac + bd}{c^2 + d^2} + i\dfrac{bc - ad}{c^2 + d^2}$

$= A + iB$, where $A = \dfrac{ac + bd}{c^2 + d^2}$ and $B = \dfrac{bc - ad}{c^2 + d^2}$

To put the complex number $\dfrac{a + ib}{c + id}$ in the form $A + iB$ we should multiply the numerator and the denominator by the conjugate of the denominator.

MODULUS OF A COMPLEX NUMBER

Modulus of a complex number $z = a + ib$ is denoted by mod(z) or $|z|$ and is defined as

$$|z| = \sqrt{a^2 + b^2}, \text{ where } a = \text{Re}(z) \text{ and } b = \text{Im}(z)$$

Sometimes, $|z|$ is called *absolute value* of z. Note that $|z| \geq 0$.

If $xy < 0$, then $\sqrt{a+ib} = \pm\left(\sqrt{\dfrac{\sqrt{a^2+b^2}+a}{2}} - i\sqrt{\dfrac{\sqrt{a^2+b^2}-a}{2}}\right)$

Thus, square roots of $z = a + ib$ are:

$$\pm\left[\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|-a}{2}}\right], \text{ for } b > 0$$

and $$\pm\left[\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|-a}{2}}\right], \text{ for } b < 0$$

Illustration 10. Let $a + bi$ be a square root of $7 + 24i$. Then

$$(a + bi)^2 = 7 + 24i \Rightarrow a^2 + 2abi + i^2b^2 = 7 + 24i$$
$$\Rightarrow (a^2 - b^2) + 2abi = 7 + 24i$$
$$\Rightarrow a^2 - b^2 = 7 \quad ...(1) \quad \text{and } 2ab = 24 \quad ...(2)$$

Now, $(a^2 + b^2)^2 = (a^2 - b^2)^2 + 4a^2b^2 = (7)^2 + (24)^2 = 49 + 576 = 625$

$\therefore$ $a^2 + b^2 = \sqrt{625} = 25$ ...(3)

Solving (1) and (3), we get

$$2a^2 = 32 \quad \text{or} \quad a^2 = 16 \quad \therefore \quad a = \pm 4$$
$$2b^2 = 18 \quad \text{or} \quad b^2 = 9 \quad \therefore \quad b = \pm 3$$

From (2), $ab = 12$ which is positive, therefore, either $a = 4, b = 3$ or $a = -4, b = -3$.

Hence the two square roots are $4 + 3i$ and $-4 - 3i$, i.e., $\pm(4 + 3i)$.

Example 1. *Express the following in the form $a + ib$, where $a, b \in R$:*

(i) $(5i)\left(\dfrac{3}{5}i\right)$; (ii) $i^9 + bi^{19}$; (iii) i^{-39}; (iv) $3(7 + i7) + i(7 + i7)$;

(v) $(1 - i) - (-1 + i6)$; (vi) $\left(\dfrac{1}{5} + i\dfrac{2}{5}\right) - \left(4 + i\dfrac{5}{2}\right)$;

(vii) $\left[\left(\dfrac{1}{3} + i\dfrac{7}{3}\right)\right] + \left(4 + i.\dfrac{1}{3}\right) - \left(-\dfrac{4}{3} + i\right)$; (viii) $(1 - i)^4$; (ix) $\left(\dfrac{1}{3} + 3i\right)^3$;

(x) $\left(-2 - \dfrac{1}{3}i\right)^3$.

Solution:

(i) $(5i)\left(-\dfrac{3i}{5}\right) = \left(5 \times \dfrac{3}{5}\right) \times (i \times i)$

$= -3i^2 = (-3)(-1) = 3$

$= a + ib$ where $a = 3, b = 0$

(ii) $i^9 + i^{19} = i \,.\, i^8 + \,.\, i^{18} = i(i^2)^4 + i(i^2)^9$

$= i(-1)^4 + i(-1)^9 = i - i = 0$

$= a + ib$ where $a = 0, b = 0$

Example 3. *Express the following in the standard form a + ib:*

(i) $\dfrac{1-2i}{2+i}+\dfrac{4-i}{3+2i}$; (ii) $(1-2i)^{-3}$; (iii) $\dfrac{2+\sqrt{-9}}{-5-\sqrt{-16}}$; (iv) $\dfrac{(1+i)^3}{4+3i}$;

(v) $(\sqrt{5}+7i)(\sqrt{5}-7i)^2+(-2+7i)^2$; (vi) $\dfrac{(3+i\sqrt{5})(3-i\sqrt{5})}{(\sqrt{3}+\sqrt{2}i)-(\sqrt{3}-i\sqrt{2})}$.

Solution:

(i) $$\frac{1-2i}{2+i}+\frac{4-i}{3+2i}=\frac{(1-2i)(3+2i)+(4-i)(2+i)}{(2+i)(3+2i)}$$

$$=\frac{3+2i-6i-4i^2+8+4i-2i-i^2}{6+4i+3i+2i^2}=\frac{16-2i}{4+7i}$$

$$=\frac{16-2i}{4+7i}\times\frac{4-7i}{4-7i}=\frac{64-112i-8i+14i^2}{16-49i^2}=\frac{50-120i}{16+49}$$

$$=\frac{50-120i}{65}=\frac{10}{13}-\frac{24}{13}i$$

(ii) $$(1-2i)^{-3}=\frac{1}{(1-2i)^3}=\frac{1}{1-8i^3-6i+12i^2}=\frac{1}{1+8i-6i-12}=\frac{1}{-11+2i}$$

$$=\frac{1}{-11+2i}\times\frac{-11-2i}{-11-2i}=\frac{-11-2i}{(-11)^2-(2i)^2}$$

$$=\frac{-11-2i}{125}=-\frac{11}{125}-\frac{2}{125}i$$

(iii) $$\frac{2+\sqrt{-9}}{-5-\sqrt{-16}}=\frac{2+3i}{-5-4i}=\frac{2+3i}{-5-4i}\times\frac{-5+4i}{-5+4i}=\frac{(2+3i)(-5+4i)}{(-5-4i)(-5+4i)}$$

$$=\frac{-10-12+i(-15+8)}{(-5)^2-(4i)^2}=\frac{-22-7i}{41}=-\frac{22}{41}-\frac{7}{41}i$$

(iv) $$\frac{(1+i)^3}{4+3i}=\frac{1^3+3\cdot1^2\cdot i+3\cdot1\cdot i^2+i^3}{4+3i}=\frac{1+3i-3-i}{4+3i}$$

$$=\frac{-2+2i}{4+3i}=\frac{(-2+2i)(4-3i)}{(4+3i)(4-3i)}=\frac{-8+8i+6i-6i^2}{4^2-9i^2}$$

$$=\frac{-2+14i}{25}=-\frac{2}{25}+i\frac{14}{25}$$

(v) $(\sqrt{5}+7i)(\sqrt{5}-7i)^2+(-2+7i)^2$

$$=(\sqrt{5}+7i)(\sqrt{5}-7i)(\sqrt{5}-7i)+[(-2)^2+(7i)^2+2(-2)(7i)]$$

$$=[(\sqrt{5})^2-(7i)^2](\sqrt{5}-7i)+(4+49i^2-28i)$$

$$=(5-49i^2)(\sqrt{5}-7i)+(4-49-28i)$$

$$=(5+49)(\sqrt{5}-7i)+(-45-28i)$$

$$=(54\sqrt{5}-378i-45-28i)=(54\sqrt{5}-45)-406i$$

Example 12. *If* $(x + iy)^{1/3} = a + ib$, *show that* $4(a^2 - b^2) = \frac{x}{a} + \frac{y}{b}$.

Solution: We have,

$$(x + iy)^{1/3} = a + ib$$

Cubing both sides,

$$x + iy = (a + ib)^3 = a^3 + (ib)^3 + 3abi\,(a + ib)$$

$$= a^3 - ib^3 + 3a^2bi - 3ab^2 \qquad [\because i^3 = i^2 i = -i]$$

$$= a\,(a^2 - 3b^2) + ib\,(3a^2 - b^2)$$

$\Rightarrow \quad x = a(a^2 - 3b^2)$ and $y = b(3a^2 - b^2)$

$\Rightarrow \quad \frac{x}{a} = a^2 - 3b^2$ and $\frac{y}{b} = 3a^2 - b^2 \qquad \therefore \quad \frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$.

Example 13. *If* $a^2 + b^2 = 1$, *prove that* $\left(\frac{1+b+ia}{1+b-ia}\right) = (b + ia)$.

Solution: We have,

$$\frac{1+b+ia}{1+b-ia} = \frac{1+b+ia}{1+b-ia} \times \frac{1+b+ia}{1+b+ia} = \frac{(1+b+ia)^2}{(1+b)^2 - (ia)^2}$$

$$= \frac{1 + b^2 - a^2 + 2b + 2ia + 2iab}{(1+b)^2 + a^2}$$

$$= \frac{2b^2 + 2b + 2ia + 2iab}{1 + (a^2 + b^2) + 2b} = \frac{2(1+b)(b+ia)}{2(1+b)}$$

$$= b + ia \qquad [\because a^2 + b^2 = 1]$$

Example 14. *If* $x + iy = \frac{a+i}{a-i}$, *prove that* $ay - 1 = x$.

Solutioin: We have,

$$x + iy = \frac{a+i}{a-i} = \frac{a+i}{a-i} \times \frac{a+i}{a+i} = \frac{(a+i)^2}{a^2 - i^2} = \frac{a^2 - 1 + 2ai}{a^2 + 1} = \frac{a^2 - 1}{a^2 + 1} + \frac{2a}{a^2 + 1} i$$

$\Rightarrow \quad x = \frac{a^2 - 1}{a^2 + 1}$ and $y = \frac{2a}{a^2 + 1}$

$\therefore \quad ay - 1 = \frac{2a^2}{a^2 + 1} - 1 = \frac{a^2 - 1}{a^2 + 1} = x$

Example 15. *If* $\frac{3}{2 + \cos\theta + i\sin\theta} = a + ib$, *prove that* $a^2 + b^2 = 4a - 3$.

Solution: Given, $\frac{2 + \cos 6 + i \sin 6}{3} = \frac{1}{a + ib} = \frac{a - ib}{a^2 + b^2}$

Example 24. *If two complex numbers z_1 and z_2 are such that $|z_1| = |z_2|$, is it then necessary that $z_1 = z_2$?*

Solution: No, it is not necessary. For example, if $z_1 = 3 + i$ and $z_2 = 1 - 3i$, then $z_1 \neq z_2$.

But $\quad |z_1| = \sqrt{(3)^2 + (1)^2} = \sqrt{9+1} = \sqrt{10}$

and $\quad |z_2| = \sqrt{(1)^2 + (-3)^2} = \sqrt{1+9} = \sqrt{10} \Rightarrow |z_1| = |z_2|$

Example 25. *If $|z| = 1$, $z \neq -1$, prove that $\frac{z-1}{z+1}$ is a purely imaginary number. What will be your conclusion if $z = 1$?*

Solution: Let $z = x + iy$ ($\neq -1$), where $|z| = 1$.

$\therefore \quad \sqrt{x^2 + y^2} = 1 \Rightarrow x^2 + y^2 = 1$

Now,
$$\frac{z-1}{z+1} = \frac{(x+iy)-1}{(x+iy)+1} = \frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}$$
$$= \frac{(x^2 - 1 + y^2) + iy(x+1-x+1)}{(x+1)^2 + y^2} = \frac{(x^2 + y^2 - 1) + 2iy}{x^2 + y^2 + 2x + 1}$$
$$= \frac{2iy}{2+2x} = 0 + i\left(\frac{y}{x+1}\right), \text{ which is purely imaginary.}$$

When $z = 1$, then $x + iy = 1 + i0 \Rightarrow x = 1$ and $y = 0$.

$$\therefore \quad \frac{z-1}{z+1} = \left(\frac{x+iy-1}{x+iy+1}\right) = \frac{1+0i-1}{1+0i+1} = 0, \text{ which is purely real.}$$

Example 26. *If $z = x + iy$, x, y real, prove that $|x| + |y| \leq \sqrt{2}|z|$.*

Solution: We have
$$(|x|-|y|)^2 \geq 0 \Rightarrow |x|^2 + |y|^2 - 2|x||y| \geq 0$$
$$\Rightarrow 2|x||y| \leq |x|^2 + |y|^2$$
$$\Rightarrow |x|^2 + |y|^2 + 2|x||y| \leq 2|x|^2 + 2|y|^2$$
$$\Rightarrow (|x|+|y|)^2 \leq 2(x^2 + y^2)$$
$$\Rightarrow (|x|+|y|)^2 \leq 2|z|^2 \qquad [\because |x|^2 = x^2 \text{ and } |y|^2 = y^2]$$
$$\therefore \quad |x| + |y| \leq \sqrt{2}|z|$$

Example 27. *If z satisfies the equation $|z| - z = 1 + 2i$, then find the value of z.*

Solution: Let $z = x + iy$.

$$\therefore \quad |z| - z = 1 + 2i \Rightarrow \sqrt{x^2 + y^2} - (x+iy) = 1 + 2i$$
$$\Rightarrow (\sqrt{x^2 + y^2} - x) + i(-y) = 1 + 2i$$
$$\Rightarrow \sqrt{x^2 + y^2} - x = 1 \text{ and } y = -2$$

If $\quad y = -2, \sqrt{x^2 + 4} - x = 1 \Rightarrow x^2 + 4 = (1+x)^2 \Rightarrow 2x = 3; x = \frac{3}{2}.$

$$\therefore \quad z = x + iy = \frac{3}{2} - 2i$$

$$2a^2 = 2 \text{ or } a^2 = 1, \qquad \therefore\ a = \pm 1$$
$$2b^2 = 6 \text{ or } b^2 = 3, \qquad \therefore\ b = \pm\sqrt{3}$$

From (2), $ab = \sqrt{3}$ which is positive,

$\therefore$ either $a = 1,\ b = \sqrt{3}$ or $a = -1,\ b = -\sqrt{3}$.

Hence the two square roots are $1 + \sqrt{3}i$ and $-1 - \sqrt{3}i$.

So $$(-2 + 2\sqrt{-3})^{1/2} = (-2 + 2\sqrt{3}i)^{1/2} = \pm(1 + \sqrt{3}i) \qquad ...(4)$$

$\therefore$ $$(-2 - 2\sqrt{-3})^{1/2} = \pm(1 - \sqrt{3}i) \qquad ...(5)$$

Adding (4) and (5), we get

$$(-2 + 2\sqrt{-3})^{1/2} + (-2 - 2\sqrt{-3})^{1/2} = \pm 2$$

Example 37. *If* $(a + ib) = \dfrac{(x+i)^2}{2x^2+1}$, *prove that* $a^2 + b^2 = \dfrac{(x^2+1)^2}{(2x^2+1)^2}$

Solution: We have,

$$(a + ib) = \frac{(x+i)^2}{2x^2+1} \qquad ...(1)$$

Replacing i by $-i$, we get

$$a - ib = \frac{(x-i)^2}{2x^2+1} \qquad ...(2)$$

Multiplying (1) and (2), we get

$$(a + ib)(a - ib) = \frac{(x+i)^2}{2x^2+1} \times \frac{(x-i)^2}{2x^2+1}$$

or $$a^2 - i^2b^2 = \frac{[(x+i)(x-i)]^2}{(2x^2+1)^2}$$

or $$a^2 - b^2 = \frac{(x^2 - i^2)^2}{(2x^2+1)^2} = \frac{(x^2+1^2)^2}{(2x^2+1)^2}$$

$\therefore$ $$a^2 + b^2 = \frac{(x^2+1^2)^2}{(2x^2+1)^2}$$

Example 38. *Find the real numbers x and y if* $(x - iy)(3 + 5i)$ *is the conjugate of* $-6 - 24i$.

Solution: Conjugate of $-6 - 24i$ is $-6 + 24i$ ...(1)

Also, $$(x - iy)(3 + 5i) = 3x - 5yi^2 - 3yi + 5xi$$
$$= (3x + 5y) + (5x - 3y)i \qquad ...(2)$$

From (1) and (2), we have

$$3x + 5y + (5x - 3y)i = -6 + 24i$$

$\therefore$ $3x + 5y = -6$ and $5x - 3y = 24$

or $3x + 5y + 6 = 0$ and $5x - 3y - 24 = 0$

$$\Rightarrow \quad \frac{x}{-120 + 18} = \frac{y}{30 + 72} = \frac{1}{-9 - 25} \quad \text{or} \quad \frac{x}{-120} = \frac{y}{102} = \frac{1}{-34}$$

30. Prove that the following complex numbers are purely real:

(i) $\left(\frac{4+3i}{3+2i}\right)\left(\frac{4-3i}{3-2i}\right)$; (ii) $\left(\frac{2+i}{3-2i}\right)+\left(\frac{2-i}{3+2i}\right)$.

31. Find the number of non-zero integral solutions of the equation $(1-i)^n = 2^n$.

32. If α and β are different complex numbers with $|\beta| = 1$, then show that $\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right| = 1$.

LEVEL OF DIFFICULTY B

33. If $z = x + iy$ and $w = \frac{1-iz}{z-i}$, show that $|w| = 1 \Rightarrow z$ is purely real.

34. If $(1-5i)z_1 - 2z_2 = (3-7i)$, find z_1 and z_2, where z_1 and z_2 are conjugate complex numbers.

35. Show that a real value of x will satisfy the equation $\frac{1-ix}{1+ix} = a - ib$ if $a^2 + b^2 = 1$, where a, b are real.

36. Prove that the sum and product of two complex numbers are real if and only if they are conjugate of each other.

37. If $\left(\frac{1-i}{1+i}\right)^{100} = a + ib$, then show that $a = 1$ and $b = 0$.

38. If z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$, then is it necessary that $z_1 = z_2$?

39. If z is a complex number such that $|z-i| = |z+i|$, show that $\text{Im}(z) = 0$.

40. Let $z_1 = a + ib$, $z_2 = c + id$ be two complex numbers such that $|z_1| = |z_2| = 1$ and $\text{Re}(z_1\overline{z_2}) = 0$. If $w_1 = a + ic$ and $w_2 = b + id$, then show that $\text{Re}(w_1\overline{w_2}) = 0$

41. For any two complex numbers z_1 and z_2 and any two real numbers a and b, show that

$$|az_1 - bz_2|^2 + |bz_1 + az_2|^2 = (a^2 + b^2)(|z_1|^2 + |z_2|^2)$$

42. Let z_1 and z_2 be complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then show that $\frac{z_1+z_2}{z_1-z_2}$ is purely imaginary.

43. If $\frac{3}{2+\cos\theta+i\sin\theta} = x + iy$, then show that $(x-1)(x-3) = -y^2$.

44. If $(a+ib)(c+id)(e+if)(g+ih) = A + iB$, then show that

$$(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2) = A^2 + B^2$$

45. For any complex number z, find the minimum value of $|z| + |z-1|$.

46. Find the maximum value of $|z|$ when z satisfies the condition $\left|z+\frac{2}{z}\right| = 2$.

47. Find the solutions of the equation $z^2 + \bar{z} = 0$.

48. Find the complex number z satisfying the equations $\left|\frac{z-12}{z-8i}\right| = \frac{5}{3}$, $\left|\frac{z-4}{z-8}\right| = 1$.

49. If z is a complex number such that $|z-1| = |z+1|$, show that $\text{Re}(z) = 0$.

34. Let $z_1 = x + iy$, then $z_2 = x - iy$. Substitute z_1 and z_2 in the given equation. Compare real and imaginary parts on both the sides and find x, y and hence z_1, z_2.

35. We have $\dfrac{1-ix}{1+ix} = \dfrac{a-ib}{1} \Rightarrow \dfrac{(1-ix)+(1+ix)}{(1-ix)-(1+ix)} = \dfrac{a-ib+1}{a-ib-1}$

[Applying componendo and dividendo]

$\Rightarrow \quad \dfrac{2}{-2ix} = \dfrac{1+a-ib}{-(1-a+ib)} \Rightarrow ix = \dfrac{1-a+ib}{1+a-ib} = \dfrac{1-a+ib}{1+a-ib} \times \dfrac{(1+a+ib)}{(1+a+ib)}$

$\Rightarrow \quad ix = \dfrac{1-a^2-b^2+2ib}{(1+a)^2+b^2} = \dfrac{2ib}{(1+a)^2+b^2}$, if $a^2 + b^2 = 1$

$\therefore \quad x = \dfrac{2b}{(1+a)^2+b^2}$, which is real.

36. Let the two complex numbers be conjugate of each other.

Let $z_1 = a + ib$, then $z_2 = a - ib$.

$\therefore$ $z_1 + z_2 = 2a$, which is real and $z_1 z_2 = a^2 + b^2$, which is real.

Conversely, let $z_1 + z_2$ and $z_1 z_2$ be both real.

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$. Then $z_1 + z_2$ and $z_1 z_2$ are real.

$\Rightarrow$ $(a_1 + a_2) + i(b_1 + b_2)$ and $(a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1)$ are real.

$\Rightarrow$ $b_1 + b_2 = 0$ and $a_1 b_2 + a_2 b_1 = 0$

$\Rightarrow$ $b_2 = -b_1$ and $a_1 b_2 + a_2 b_1 = 0 \Rightarrow b_2 = -b_1$ and $-a_1 b_1 + a_2 b_1 = 0$

$\Rightarrow$ $b_2 = -b_1$ and $(a_2 - a_1)\, b_1 = 0 \Rightarrow b_2 = -b_1$ and $a_2 = a_1$

$\Rightarrow$ $z_2 = a_2 + ib_2 = a_1 - ib_1 \Rightarrow z_2 = \overline{z}_1$.

$\Rightarrow$ z_1 and z_2 are conjugate of each other.

37. $\left(\dfrac{1-i}{1+i}\right)^{100} = (-i)^{100} = [(-i)^4]^{25} = 1 \qquad \therefore \; a + ib = 1 \Rightarrow a = 1$ and $b = 0$

38. If $z_1 = 2 + 3i$ and $z_2 = 3 + 2i$ then $|z_1| = \sqrt{4+9} = \sqrt{13}$ and $|z_2| = \sqrt{9+4} = \sqrt{13}$.

This shows that $|z_1| = |z_2|$, but $z_1 \neq z_2$.

40. $|z_1| = |z_2| = 1 \Rightarrow a^2 + b^2 = c^2 + d^2 = 1$ ...(1)

$\text{Re}(z_1 \overline{z_2}) = ac + bd = 0 \Rightarrow \dfrac{a}{b} = \dfrac{d}{-c} = \lambda$ (say) ...(2)

$\text{Re}(w_1 \overline{w_2}) = \text{Re}(a + ic)(b - id) = ab + cd$

From (1) and (2), $b^2(1 + \lambda^2) = d^2\left(1 + \dfrac{1}{\lambda^2}\right) = 1 \Rightarrow \dfrac{b^2}{d^2} = \dfrac{1}{\lambda^2}$ or $d^2 = b^2\lambda^2$

$\therefore$ $\text{Re}(w_1 \overline{w_2}) = b^2\lambda - \dfrac{d^2}{\lambda} = b^2\lambda - b^2\lambda = 0$

41. $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$

$= (az_1 - bz_2)(a\overline{z_1} - b\overline{z_2}) + (bz_1 + az_2)(b\overline{z_1} + a\overline{z_2})$

$= a^2 z_1 \overline{z_1} + b^2 z_2 \overline{z_2} - abz_1 \overline{z_2} - abz_2 \overline{z_1} + b^2 z_1 \overline{z_1} + a^2 z_2 \overline{z_2} + abz_1 \overline{z_2} + abz_2 \overline{z_1}$

$= (a^2 + b^2)(|z_1|^2 + |z_2|^2)$ $[\because z\overline{z} = |z|^2]$

42. Let $z_1 = a + ib$ and $z_2 = c - id$, where a and d are positive real quantities.

$\therefore$ $|z_1| = |z_2| \Rightarrow a^2 + b^2 = c^2 + d^2$ ...(1)

Draw $PM \perp OX$ as shown in Fig. 5.2. Then, $OM = x$ and $PM = y$. Join OP. Let $OP = r$ and $\angle XOP = \theta$. Then $x = r\cos\theta$ and $y = r\sin\theta$.

$$\therefore \quad z = x + iy = r(\cos\theta + i\sin\theta)$$

Comparing real and imaginary parts, we get

$$x = r\cos\theta \quad \text{...(1)}$$

and $$y = r\sin\theta \quad \text{...(2)}$$

Squaring (1) and (2) and adding, we get

$$r^2 = x^2 + y^2 \text{ or } r = \sqrt{x^2 + y^2} = |z|$$

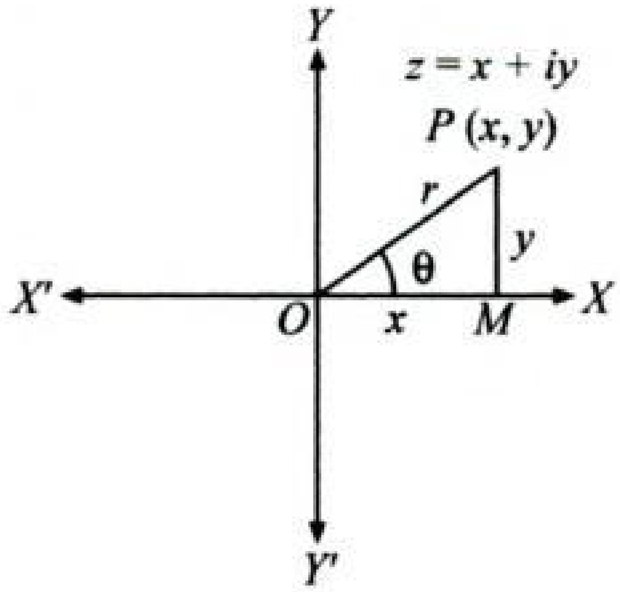

Fig. 5.2

Also $$\tan\theta = \frac{y}{x}$$

This form $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ is called *polar form* of the complex number z. Angle θ is known as *amplitude* or *agrument* of z, written as arg(z).

The unique value of θ such that $-\pi < \theta \le \pi$ for which $x = r\cos\theta$ and $y = r\sin\theta$, is known as the *principal value of the argument.*

The general value of the argument is $(2n\pi + \theta)$, where n is an integer and θ is the principal value of arg(z).

While reducing a complex number to polar form, we always take the principal value.

Trick(s) for Problem Solving

- If $x > 0, y > 0$ (i.e., z is in first quadrant), then
$$\arg(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right)$$
- If $x < 0, y > 0$ (i.e., z is in second quadrant), then
$$\arg(z) = \theta = \pi - \tan^{-1}\left(\frac{y}{|x|}\right)$$
- If $x < 0, y < 0$ (i.e., z is in third quadrant), then
$$\arg(z) = \theta = -\pi + \tan^{-1}\left(\frac{y}{x}\right)$$
- If $x > 0, y < 0$ (i.e., z is in fourth quadrant), then
$$\arg(z) = \theta = -\tan^{-1}\left(\frac{|y|}{x}\right)$$
- Argument of the complex number 0 is not defined.
- $\arg(x + i0) = \begin{cases} 0, \text{ if } x > 0 \\ \pi, \text{ if } x < 0 \end{cases}$ • $\arg(0 + iy) = \begin{cases} \frac{\pi}{2}, \text{ if } y > 0 \\ \frac{3\pi}{2}, \text{ if } y < 0 \end{cases}$.

Example 1. *Write the following complex numbers in polar form and determine the modulus and the principal value of the agrument in each case:*

(i) $1 + i$; (ii) $-1 + \sqrt{3}i$; (iii) $-\sqrt{3} - i$; (iv) $1 - i$;
(v) -3; (vi) $-2i$; (vii) $3 + 4i$.

(iii) $\sin 135° - i \cos 135° = \sin (90° + 45°) - i \cos (90° + 45°)$

$= \cos 45° + i \sin 45° = 1 (\cos 45° + i \sin 45°)$

(iv) $3 (\cos 300° - i \sin 30°) = 3[\cos (360° - 60°) - i \sin 30°]$

$= 3(\cos 60° - i \sin 30°) = \frac{3}{2}(1 - i).$

Let $z = 1 - i = r (\cos \theta + i \sin \theta)$, then $1 = r \cos \theta$ and $-1 = r \sin \theta$.

$\therefore \quad r^2 = 2 \quad \text{or} \quad r = \sqrt{2}.$

$\therefore \quad \cos \theta = \frac{1}{\sqrt{2}}$ (+ve)

and $\sin \theta = -\frac{1}{\sqrt{2}}$ (– ve) so that θ lies in fourth quadrant.

$$\therefore \quad \theta = -\tan^{-1}\left(\frac{1}{1}\right) = -\tan^{-1} 1 = -\frac{\pi}{4}.$$

$\therefore$ The polar form of z is $\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right)\right]$, i.e., $\sqrt{2}\left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4}\right)$.

$$\therefore \quad 3 (\cos 300° - i \sin 30°) = \frac{3}{2} \cdot \sqrt{2}\left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4}\right) = \frac{3}{\sqrt{2}}\left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4}\right)$$

Example 4. *Find the modulus and argument of the complex number* $\frac{1+2i}{1-3i}$.

Solution: $z = \frac{1+2i}{1-3i} = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i} = \frac{1+6i^2+2i+3i}{1-9i^2}$

$$= \frac{1-6+5i}{1+9} = \frac{-5+5i}{10} = \frac{-5}{10} + \frac{5}{10}i = -\frac{1}{2} + \frac{1}{2}i$$

Put $r \cos\theta = \frac{1}{2}$, and $r \sin\theta = \frac{1}{2}$,

$$\therefore \quad r^2 = \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \quad \text{or} \quad r = \frac{1}{\sqrt{2}}$$

Also, $\tan \theta = -1$

$\Rightarrow \sin \theta$ is +ve and $\cos \theta$ is –ve. $\quad \therefore \theta$ lies in second quadrant

$$\therefore \quad \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$\therefore \quad \left|\frac{1+2i}{1-3i}\right| = \frac{1}{\sqrt{2}} \text{ and } \arg\left(\frac{1+2i}{1-3i}\right) = \frac{3\pi}{4}$$

Example 5. *Find the modulus of* $\frac{1+i}{1-i} - \frac{1-i}{1+i}$.

Solution: $\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i} = \frac{(1+i)^2}{1-i^2} = \frac{1+i^2+2i}{1+1} = \frac{2i}{2} = i$

$$\frac{1-i}{1+i} = \frac{1-i}{1+i} \times \frac{1-i}{1-i} = \frac{(1-i)^2}{1-i^2} = \frac{1+i^2+2i}{2} = -i$$

$$\frac{1+i}{1-i} = \frac{1-i}{1+i} = i - (-i) = 2i$$

IMPORTANT

- To represent the quadratic $ax^2 + bx + c$ in the form (2) or (3) is to expand it into linear factors.
- If $a, b, c \in Q$ and D is a perfect square of a rational number, then the roots are rational. and In case it is not a perfect square, then the roots are irrational.
- If $a, b, c \in R$ and $p + iq$ is one root of (1), then the other root must be the conjugate $p - iq$ and vice-versa. ($p, q \in R, q \neq 0$ and $i = \sqrt{-1}$).
- If $a, b, c \in Q$ and $p + \sqrt{q}$ is one root of (1), then the other must be the conjugate $p - \sqrt{q}$ and vice-versa (where p is a rational and $\sqrt{q}$ is a surd).
- If $a = 1$ and $b, c \in I$ and the roots of equation (1) are rational numbers, then these roots must be integers.
- If quadratic equation (1) has more than two distinct roots, then (1) becomes an identity in x. i.e., $a = b = c = 0$.

REMEMBER
A polynomial equation of degree n has n roots.

Example 1. *Solve the following quadratic equations:*

(i) $x^2 + 4 = 0$; (ii) $9x^2 + 16 = 0$;

(iii) $2x^2 - 4x + 3 = 0$; (iv) $(y + 1)(y - 3) + 7 = 0$;

(v) $\sqrt{2}x^2 + x + \sqrt{2} = 0$.

Solution: (i) We have,

$$x^2 + 4 = 0 \Rightarrow x^2 - 4i^2 = 0 \Rightarrow x^2 - (2i)^2 = 0$$
$$\Rightarrow (x - 2i)(x + 2i) = 0 \Rightarrow x - 2i = 0 \text{ or } x + 2i = 0$$
$$\Rightarrow x = 2i \text{ or } x = -2i$$

Hence, the roots are $2i$ and $-2i$

(ii) We have,

$$9x^2 + 16 = 0 \Rightarrow 9x^2 - 16i^2 = 0 \Rightarrow (3x)^2 - (4i)^2 = 0$$
$$\Rightarrow (3x - 4i)(3x + 4i) = 0 \Rightarrow 3x - 4i = 0 \text{ or } 3x + 4i = 0$$
$$\Rightarrow x = \frac{4}{3}i \text{ or } x = -\frac{4}{3}i$$

(iii) We have

$$2x^2 - 4x + 3 = 0$$

Here $a = 2, b = -4, c = 3$.

$$\therefore \quad D = b^2 - 4ac = (-4)^2 - 4(2)(3) = 16 - 24 = -8 < 0$$

The equation has no real roots.

$$\therefore \quad x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-4) \pm \sqrt{-8}}{2(2)} = \frac{4 \pm \sqrt{8}i}{4} = \frac{4 \pm 2\sqrt{2}i}{4} = 1 \pm \frac{1}{\sqrt{2}}i$$

Hence, the roots are $1 + \frac{1}{\sqrt{2}}i$ and $1 - \frac{1}{\sqrt{2}}i$

(iii) The given inequality is $3(x - 1) \leq 2(x - 3)$.

On simplifying

$$3x - 3 \leq 2x - 6$$

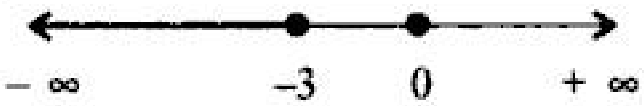

Fig. 6.3

Transposing $2x$ to L.H.S. and -3 to R.H.S.,

$$3x - 2x \leq -6 + 3 \quad \text{or} \quad x \leq -3$$

The solution is $(-\infty, -3]$ (Fig. 6.3).

(iv) The given inequality is $3(2 - x) \geq 2(1 - x)$.

Simplifying,

$$6 - 3x \geq 2 - 2x$$

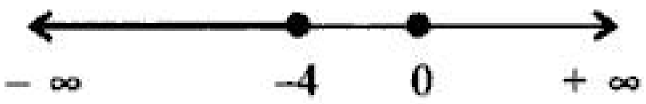

Fig. 6.4

Transposing $-2x$ to L.H.S. and 6 to R.H.S.,

$$-3x + 2x \geq 2 - 6 \quad \text{or} \quad -x \geq -4$$

Multiplying by -1, we get $x \leq 4$.

The solution is $(-\infty, 4]$ (Fig. 6.4).

(v) The given inequality is $x + \frac{x}{2} + \frac{x}{3} < 11$.

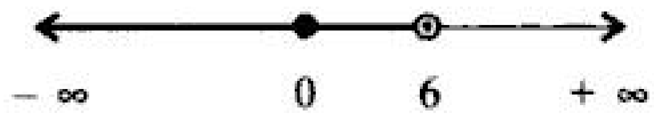

Fig. 6.5

Simplifying,

$$\frac{6x + 3x + 2x}{6} < 11 \quad \text{or} \quad \frac{11x}{6} < 11$$

Dividing both sides by $\frac{6}{11}$, we get $x < 6$.

The solution is $(-\infty, 6)$ (Fig. 6.5).

(vi) The given inequality is $\frac{x}{3} > \frac{x}{2} + 1$.

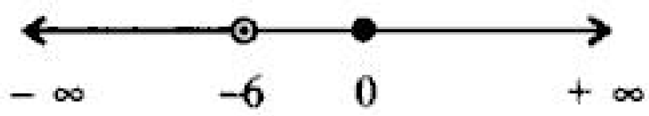

Fig. 6.6

Transposing $\frac{x}{2}$ to L.H.S., we get $\frac{x}{3} - \frac{x}{2} > 1$ or $\frac{2x - 3x}{6} > 1$ or $-\frac{x}{6} > 1$

Multiplying by -6, we get $x < -6$.

The solution is $(-\infty, -6)$ (Fig. 6.6).

(vii) The given inequality is $37 - (3x + 5) \geq 9x - 8(x - 3)$.

Simplifying

$$37 - 3x - 5 \geq 9x - 8x + 24$$

or $\quad 32 - 3x \geq x + 24$

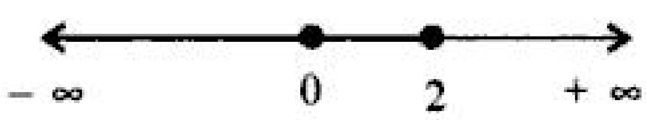

Fig. 6.7

Transposing x to L.H.S. and 32 to R.H.S., we get

$$-3x - x \geq 24 - 32 \quad \text{or} \quad -4x \geq -8$$

Dividing by -4, we get $\quad x \leq 2$.

The solutions is $(-\infty, 2]$ (Fig. 6.7).

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The line passes through the points (0, 3) and (5, 0), The graph of $3x + 5y < 15$ is the line AB.

Step 3. Putting $x = 0$, $y = 0$ in $3x + 5y < 15$, we get $0 < 15$, which is true.

Step 4. Region containing the point (0, 0) is the solution set of the given inequation. We shade the portion which contain the origin i.e., the point (0, 0). (Refer Fig. 6.27)

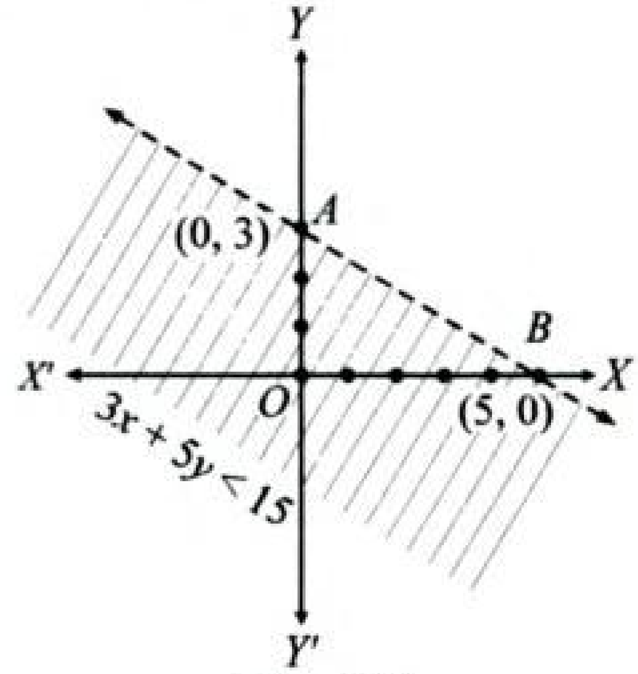

Fig. 6.27

All points in the shaded region satisfy the given inequation and form the solution set. But the points on the graph line are not included in the solution set. That is why graph line is shown dotted.

Example 3. *Solve $4x - y > 0$ graphically.*

Solution: Replacing the inequality sign by equality, we get the equation

$$4x - y = 0 \quad ...(1)$$

Putting $x = 1$, we get $y = 4$. Putting $y = 0$, we get $x = 0$.

The equation $4x - y = 0$ passes through the points $O(0, 0)$ and $A(1, 4)$. The graph of $4x - y = 0$ is the dotted line OA as shown in Fig. 6.28.

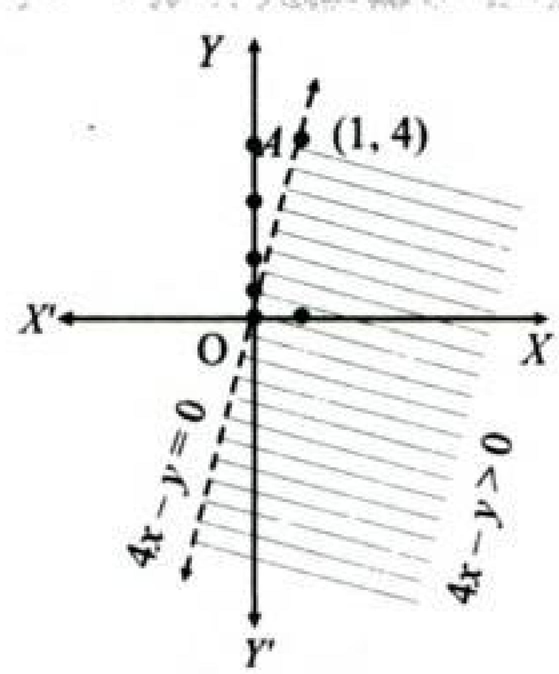

Fig. 6.28

Putting $x = 1$ and $y = 0$ in the given inequation $4x - y > 0$, we get $4 > 0$, which is true. So the region containing the point (1, 0) is the solution set of the given inequation. We shade the portion which contains the point (1, 0). All points in the shaded region satisfy the given inequality and form the solution set. But the points on the graph line are not included in the solution set. That is why graph line is shown dotted.

Example 4. *Solve the following inequations graphically:*

(i) $2x - 3 \geq 0$; (ii) $y < 3$; (iii) $x > -3$.

Solution: (i) Replacing the inequality sign by equality,

we get the equation $2x - 3 = 0$ or $x = \frac{3}{2}$, which represents a straight line parallel to y-axis at a distance $\frac{3}{2}$ units to the right of y-axis. We draw the line $x = \frac{3}{2}$. This line divides the xy-plane into two parts—one part on the L.H.S. of $x = \frac{3}{2}$ and the other on its R.H.S. Putting $x = 0$ in the

13.

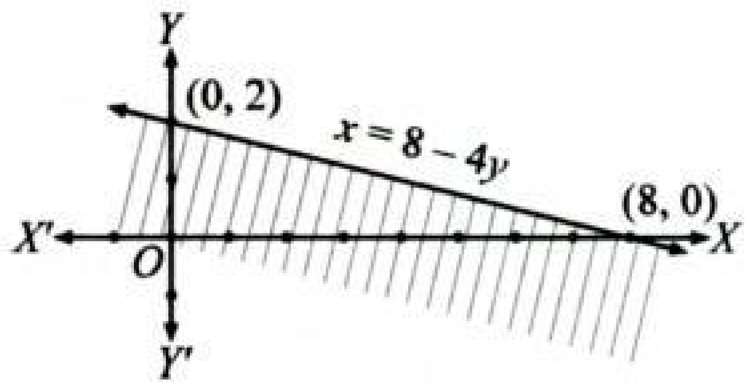

Fig. 6.44

14.

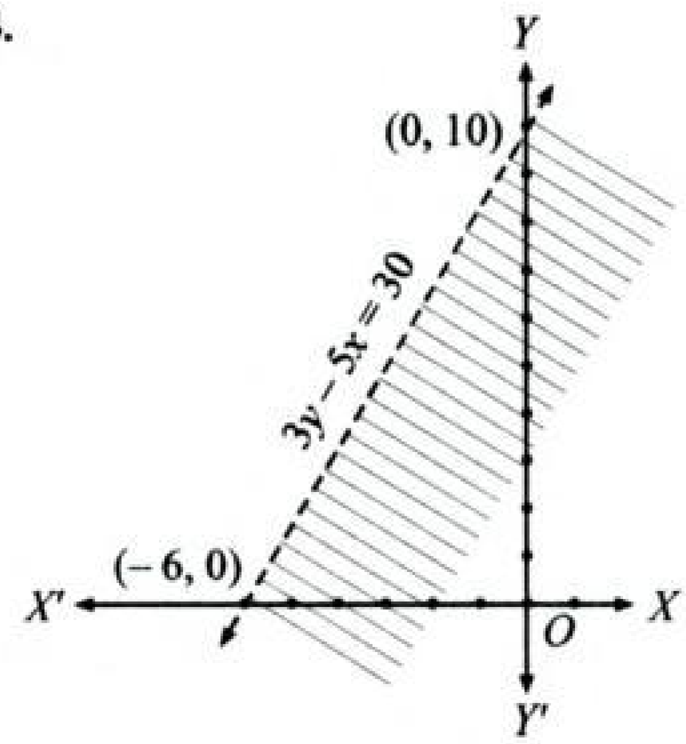

Fig. 6.45

15.

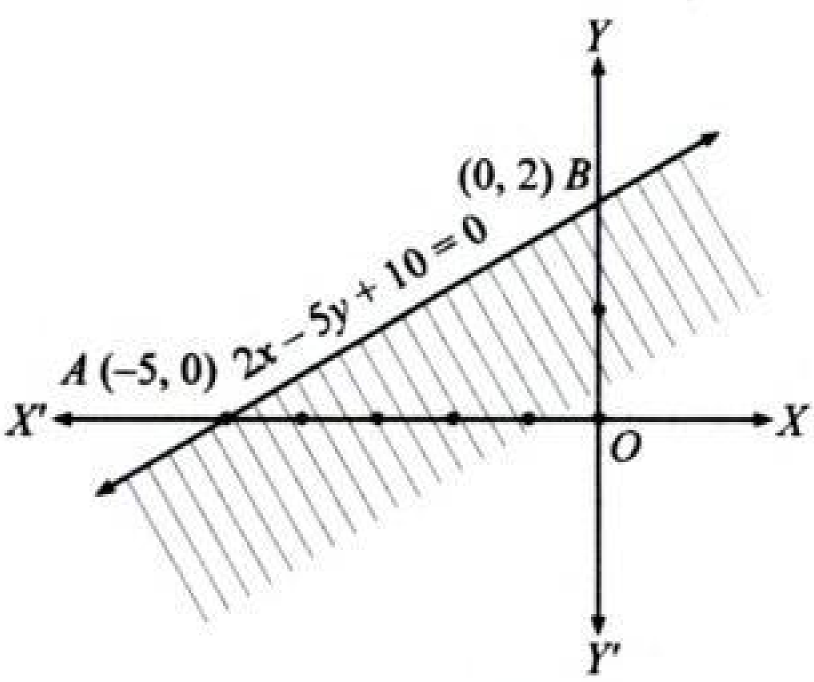

Fig. 6.46

16.

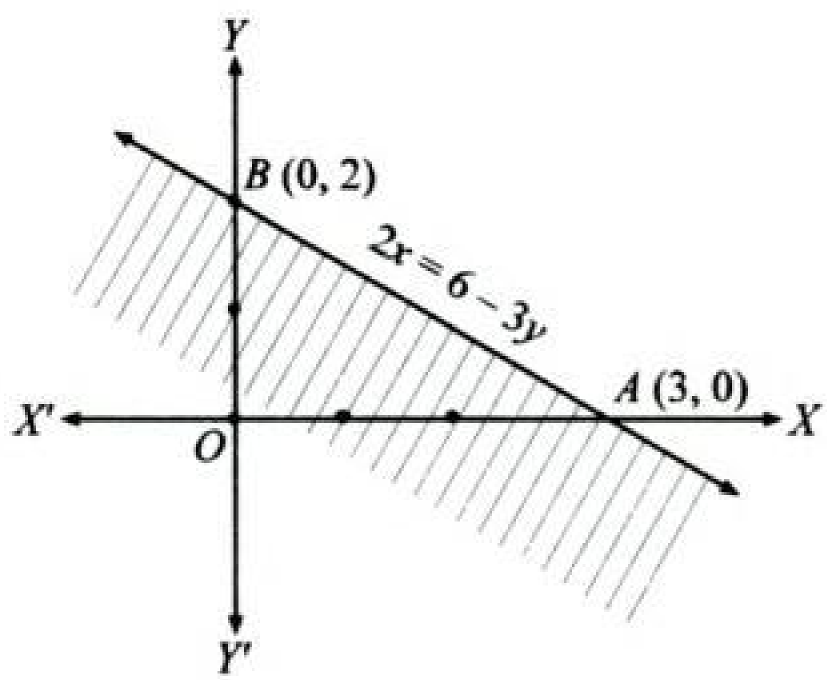

Fig. 6.47

SOLUTION OF SYSTEM OF LINEAR INEQUATIONS IN TWO VARIABLES

To solve a system of linear inequations in two variables, we proceed as follows:

(i) Draw the graphs of all the given linear inequations.

(ii) Find the common part of the coordinate plane which satisfies all the given linear inequations.

(iii) This common part of the coordinate plane is the required solution of the given inequations.

(iv) If there is no region common to all the solutions of the given inequations, we say that the solution set of the system of inequations is empty.

Note: It may be noted that the solution set of simultaneous linear inequations may be an empty set or it may be the region bounded by the straight lines corresponding to given linear inequations or it may be an unbounded region with straight line boundaries.

The following examples illustrate the above procedure:

Example 1. *Solve graphically:*

$$3x + 4y \geq 12, x \geq 1, y \geq 2$$

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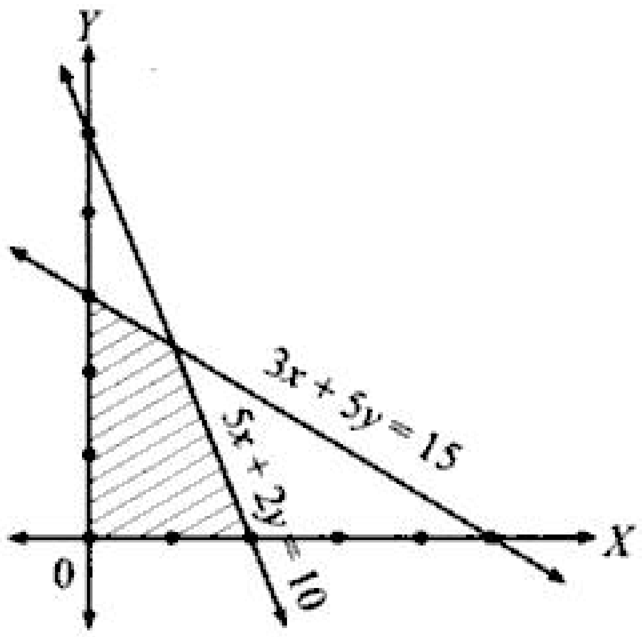

Fig. 6.69

10.

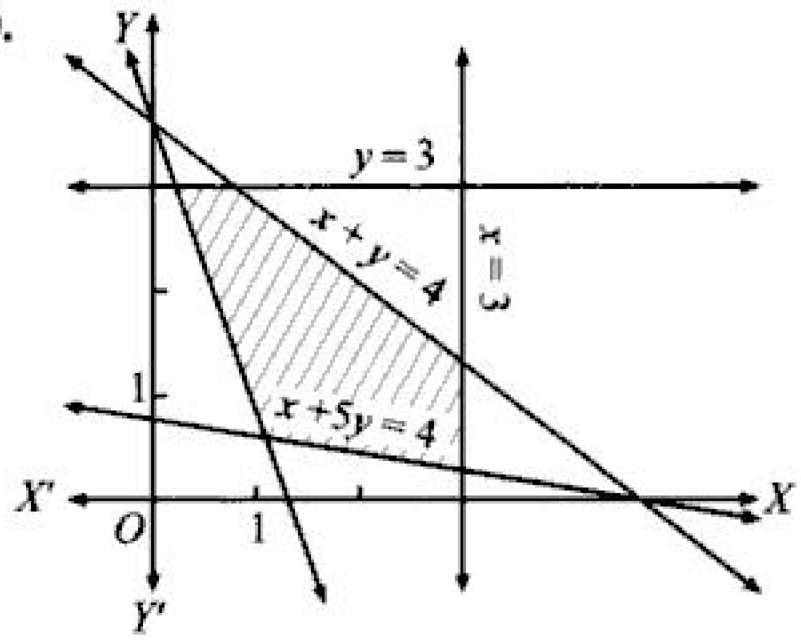

Fig. 6.70

11.

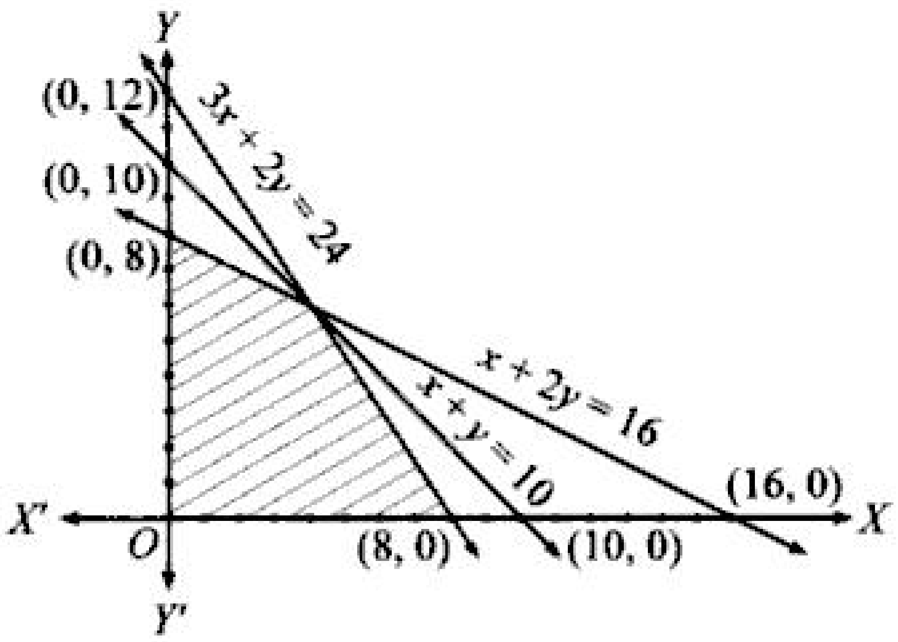

Fig. 6.71

12.

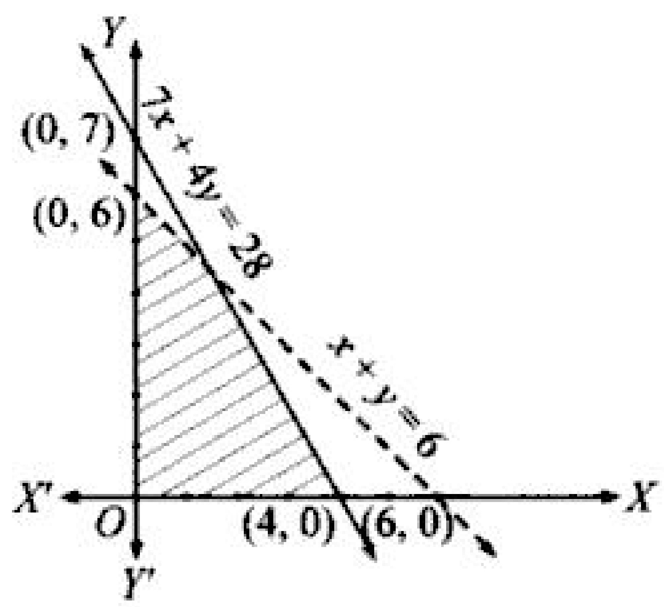

Fig. 6.72

13.

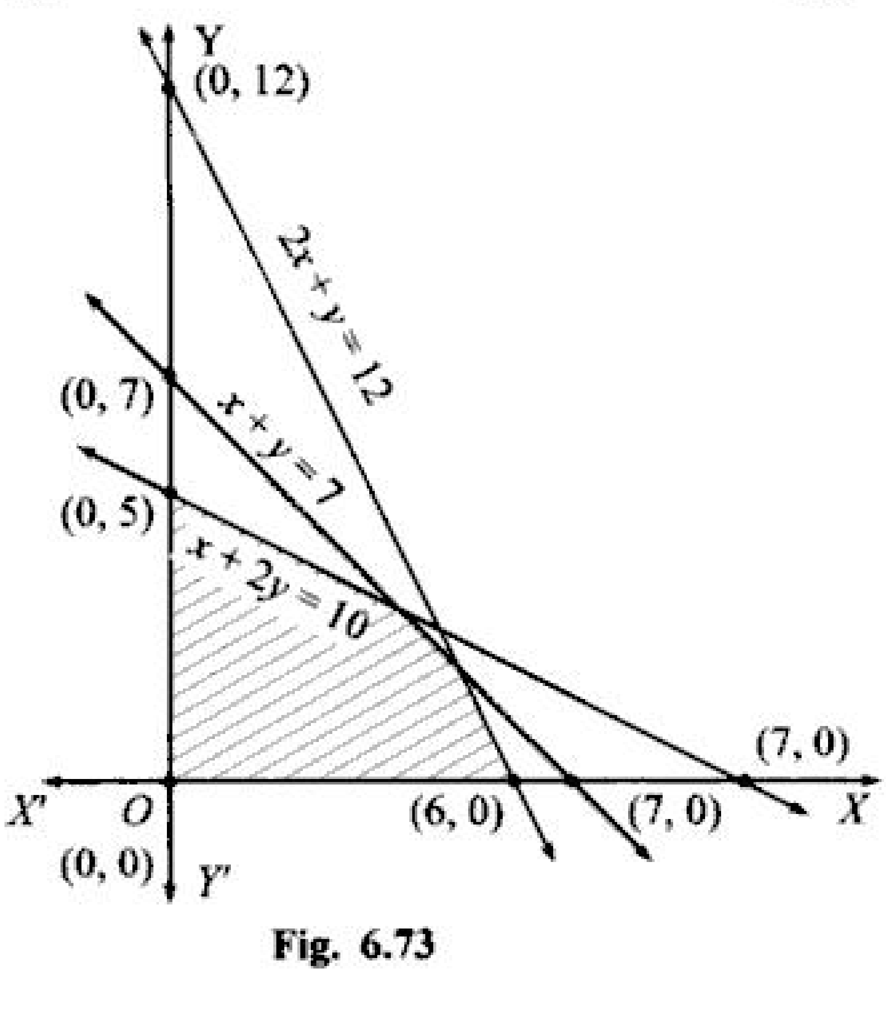

Fig. 6.73

14.

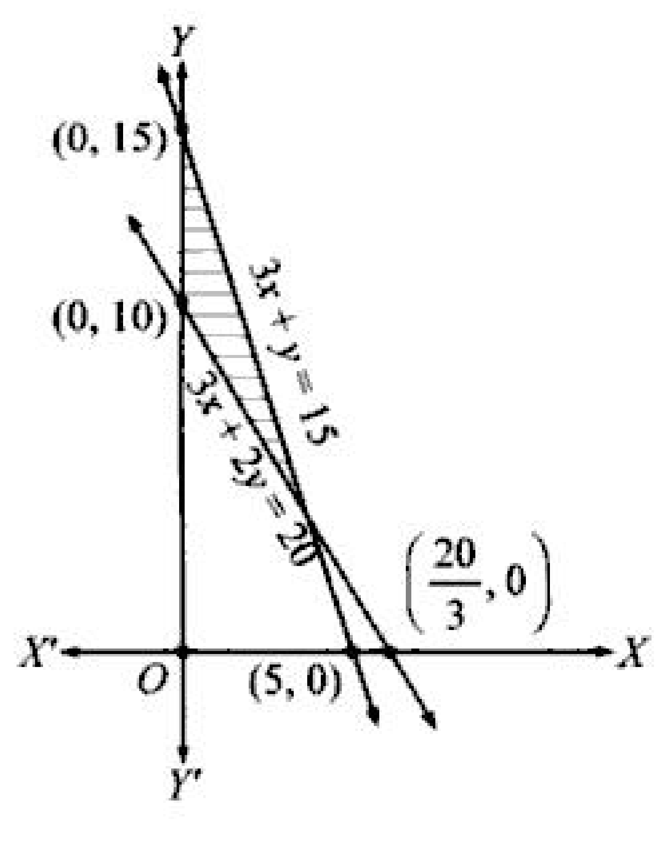

Fig. 6.74

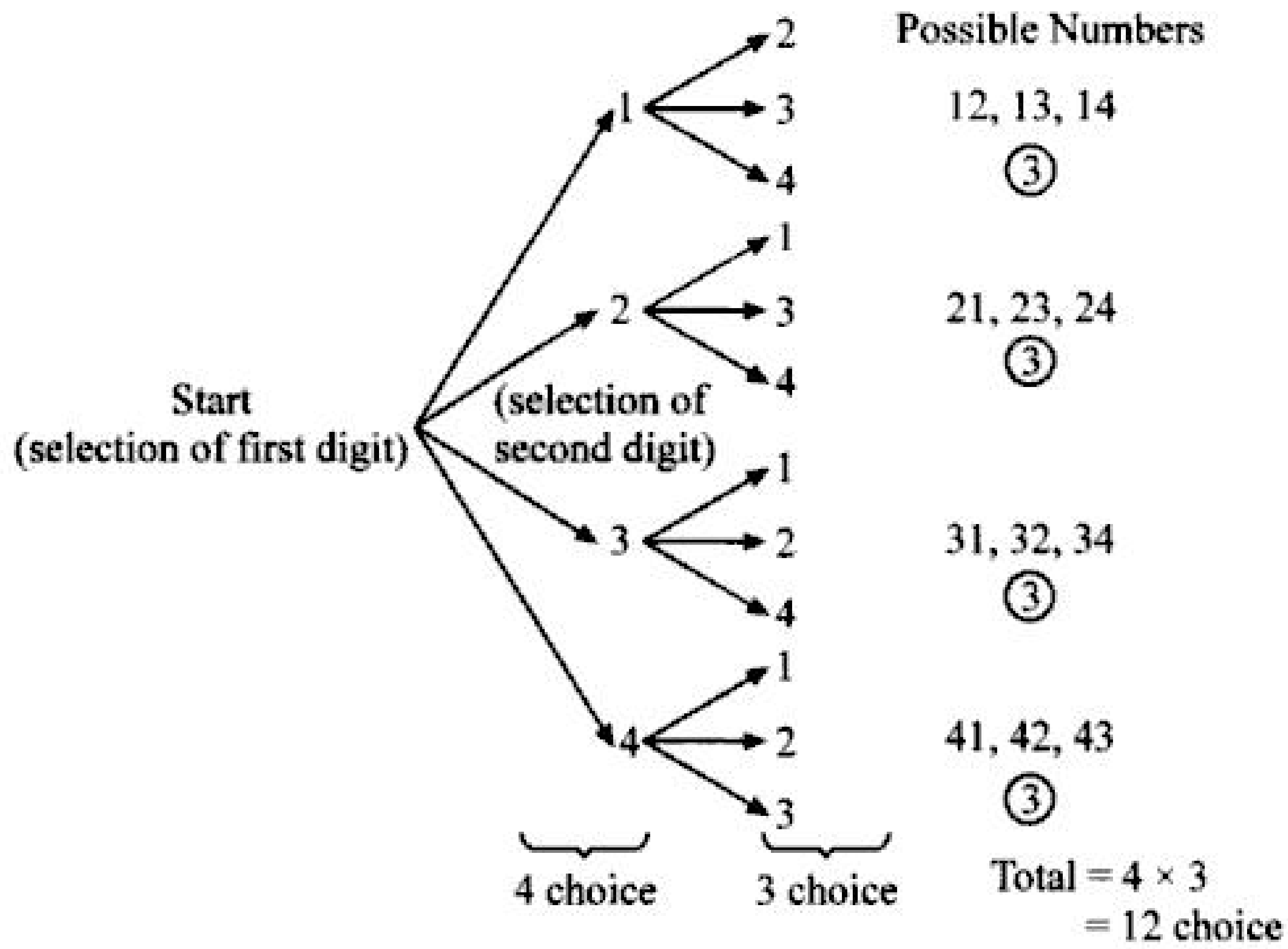

Fig. 7.1

Since the first digit can be chosen in four ways and for each choice of the first digit there are three ways of choosing the second digit, therefore, there are 4 × 3 ways, i.e., 12 ways of choosing both the digits. Thus, 12 numbers can be formed.

Illustration 2. Anu wishes to buy a birthday card for the brother manu and send it by post. Five different types of cards are available at the card-shop, and four different types of postage stamps are available at the post-office. In how many ways can she choose the card and the stamp?

Solution: She can choose the card in five ways. For each choice of the card she has four choices for the stamp. Therefore, there are 5 × 4 ways, i.e., 20 ways of choosing the card and the stamp.

Illustration 3. Mohan wishes to go from Agra to Chennai by train and return from Chennai to Delhi by air. There are six different trains from Agra to Chennai and five different flights from Chennai to Agra. In how many ways can he perform the journey?

Solution: Since he can choose any one of the six trains for going to Chennai, and for each such choice he has five choices for returning to Agra, he can perform the journey in 6 × 5 ways, i.e., 30 ways.

The above illustrations suggest that if one operation can be performed independently in m different ways and another operation can be performed independently in n different ways, then the number of ways in which both the operations can be performed in succession is mn.

The above principle can be generalized to the case of three or more operations. We thus have the following **fundamental principle of counting:**

If one operation can be performed independently in m_1 *different ways, and if a second operation can be performed in* m_2 *different ways, and a third operation can be performed in* m_3 *different ways and so on for any finite number of operations, then the total number of ways in which all the operations can be performed in the stated order is* $m_1 m_2 m_3 \ldots$

Solution: First question can be answered in 3 ways.
Second question can be answered in 3 ways.
Third question can be answered in 4 ways.
Fourth question can be answered in 4 ways.
Fifth question can be answered in 5 ways.
Sixth question can be answered in 5 ways.

Hence by fundamental principle of counting, the required number of sequences of answers

$$= 3 \times 3 \times 4 \times 4 \times 5 \times 5 = 3600$$

Example 14. *Each section in the first year of plus two course has exactly 40 students. If there are 5 sections, in how many ways can a set of 4 student representatives be selected, one from each section?*

Solution: One student representative can be selected from section I in 40 ways.
One student representative can be selected from section II in 40 ways.
One student representative can be selected from section III in 40 ways.
One student representative can be selected from section IV in 40 ways.

Hence the number of ways in which a set of 4 students representatives can be selected

$$= 40 \times 40 \times 40 \times 40 = 2560000$$

Example 15. *A team consists of 6 boys and 4 girls and the other has 5 boys and 3 girls. How many singles matches can be arranged between the two teams when a boy plays against a boy and a girl plays against a girl?*

Solution: A boy can be selected from the first team in 6 ways, and from the second in 5 ways.

The number of singles matches between boys of two teams = $6 \times 5 = 30$ Similarly, number of singles matches between girls of two teams = $4 \times 3 = 12$

Total number of singles matches = $30 + 12 = 42$.

Example 16. *In how many ways can this diagram be coloured subjected to the following two conditions?*

(i) *Each of the smaller triangles is to be painted with one of the three colours, red, blue or green.*

(ii) *No two adjacent regions receive the same colour.*

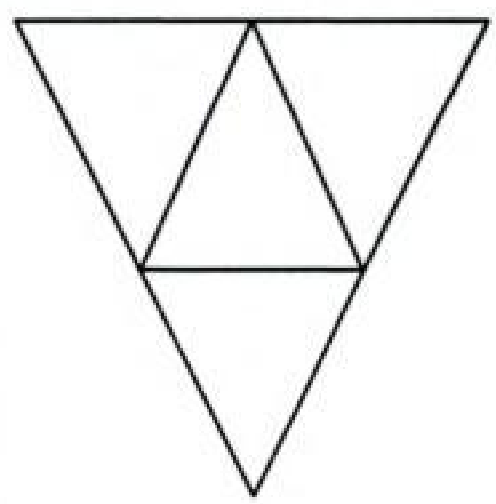

Fig. 7.3

Solution: Paint the central triangle with any one of three colours. This can be done in 3 ways. After that paint the remaining three triangles with any one of the remaining two colours.

By the fundamental principle of counting, the given diagram can be coloured in $3 \times 2 \times 2 \times 2 = 24$ ways.

Example 17. *There are 5 letters and 5 directed envelopes. Find the number of ways in which the letters can be put into the envelopes so that all are not put in directed envelopes.*

Solution: Here the first letter can be put in any one of the 5 envelopes in 5 ways. Second letter can be put in any one of the 4 remaining envelopes in 4 ways. Continuing in this way, we get the total number of ways in which 5 letters can be put into 5 envelopes

$$= 5 \times 4 \times 3 \times 2 \times 1 = 120$$

Since out of the 120 ways, there is only one way for putting each letter in the correct envelope. Hence the number of ways of putting letters all not in directed envelopes

$$= 120 - 1 = 119 \text{ ways.}$$

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(ii) For 2-digit odd numbers, unit's place can be filled up with 3 or 5, i.e., in 2 ways and ten's place can be filled up with 2 or 3 or 5, i.e., in 3 ways.

$\therefore$ 2-digit odd numbers = $2 \times 3 = 6$

(iii) For 3-digit odd numbers, unit's place can be filled up in 2 ways. Ten's place can be filled up in 4 ways. Hundred's place can be filled up in 3 ways.

$\therefore$ 3-digit odd numbers = $2 \times 4 \times 3 = 24$

(iv) For 4-digit odd numbers, unit's place can be filled up in 2 ways. Ten's place can be filled up in 4 ways. Hundred's place can be filled up in 4 ways. Thousand's place can be filled up in 3 ways.

$\therefore$ 4-digit odd numbers = $2 \times 4 \times 4 \times 3 = 96$

Example 30. *How many numbers are there between* 100 *and* 10,000 *in which all the digits are distinct?*

Solution: Any number between 100 and 10,000 is of three digits. Since the numbers should have distinct digits, therefore, repetition of digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 is not allowed.

Also 0 cannot be placed on the extreme left place. Hundredth place can be filled in 9 ways. Tenth place can be filled in 9 ways. Unit's place can be filled in 8 ways.

$\therefore$ Total three digit numbers = $9 \times 9 \times 8 = 648$

Example 31. *How many integers between* 1000 *and* 10,000 *have no digits other than* 4, 5 *or* 6?

Solution: Any number between 1000 and 10,000 is of 4 digits.

The unit's place can be filled up by 4 or 5 or 6, i.e., in 3 ways. Similarly, the ten's place can be filled up by 4 or 5 or 6, i.e., in 3 ways. The hundred's place can be filled up by 4 or 5 or 6, i.e., in 3 ways and the thousand's place can be filled up by 4 or 5 or 6, i.e., in 3 ways.

Hence Required no. of numbers = $3 \times 3 \times 3 \times 3 = 81$

Example 32. *How many numbers are there between* 100 *and* 1000 *which have exactly one of their digits as* 7?

Solution: The numbers between 100 and 1000 having 7 as exactly one of their digits can be classified into three types:

(i) **When the unit's place has 7.**

Here the ten's place can have any one of the digits except 7. It can be filled in 9 different ways. The hundred's place can have any one of the digits except 0 and 7. So hundred's place can be filled in 8 different ways. Therefore, there are $9 \times 8 = 72$ such numbers.

(ii) **When the ten's place has 7.**

The unit's place can be filled in 9 different ways. It can have any one of the digits except 7. The hundred's place can have any one of the digits except 0 and 7. So hundred's place can be filled in 8 different ways. So, there are $9 \times 8 = 72$ such numbers.

(iii) **When the hundred's place has 7.**

Here the unit's place can be filled by 9 different ways (except by 7) and ten's place can be filled by 9 different ways (except by 7). So there are $9 \times 9 = 81$ such numbers.

Required number of numbers = $72 + 72 + 81 = 225$

Example 33. *How many numbers of six-digit numbers can be formed from the digits* 0, 1, 3, 5, 7 *and* 9, *when no digit is repeated? How many of them are divisible by* 10?

35. How many 2-digit even numbers can be formed from the digits 3, 4, 5, 6 and 7, if
(i) repetition of digits is not allowed. (ii) repetition of digits is allowed.

36. There are 5 roads leading to a town from a village. Find the number of different ways in which a villager can go to the town and return back.

37. There are 5 routes from place *A* to place *B* and 3 routes from plance *B* to place *C*. Find how many different routes are there from A to *C* via *B*.

38. John wants to go abroad by ship and return by air. He has a choice of 6 different ships to go and 4 airlines to return. In how many ways can he perform his journey?

39. If there are 20 steamers playing between plances *A* and *B* in how many ways could the round trip from *A* be made if the return was made by (i) the same steamer, (ii) a different steamer.

40. In how many ways can 5 persons draw water from 5 taps, assuming no tap remains unused ?

41. Eight children are to be seated on a bench.
(i) In how many ways can the children be seated?
(ii) How many arrangements are possible if the youngest child sits at the left hand end to the bench?

42. How many 4-letter code works are possible using the first 10 letters of the english alphabet, if no letter can be repeated?

43. Six pictures are to be arranged from left to right on a wall of an art gallery for display. How many arrangements are possible?

44. The digits, from 0 to 9, written on slips of paper and placed in a box. Three of the slips of paper are drawn and placed in order. How many different outcomes are possible?

45. For a group photograph, 3 boys and two girls stand in a line in all possible ways. How many photots could be taken if each photo corresponds to each such arrangement?

46. A sample of 3 bulbs is tested. *A* bulb is labelled as '*G*' If it is good and '*D*' if it is defective. Find the number of all the possible outcomes.

47. How many 5-digit telephone numbers can be constructed using the digits 0 to 9 if each number starts with 67, for example 67125 etc. and no digit appears more than once?

48. A coin is tossed three times and the outcomes are recorded. How many possible outcomes are there? How many possible outcomes are there if the coin is tossed four times? Five times? *n* times?

49. Find the number of different signals that can be made by arranging at least three flags in order on a vertical pole, if six different flags are available.

50. Manu goes to a movie. The cinema hall has two entrances andthree exits. In how many ways can Manu enter adn exit from the hall?

Answers

1. 36.	**2.** 6.	**3.** 28.	**4.** 72.
5. 80.	**6.** 375.	**7.** (i) 125; (ii) 60	**8.** 108.
9. 1050	**10.** 120.	**11.** 60.	**12.** 120; 119.
13. 9×10^6.	**14.** (i) 90; (ii) 100.	**15.** 64.	**16.** 54.
17. 12.	**18.** 336.	**19.** (i) 156; (ii) 1920.	
20. 180.	**21.** 4.	**22.** 4536.	**23.** 4536.
24. 14.	**25.** 1630.	**26.** 42.	**27.** $2^n - 2$.
28. 56.	**29.** 9000.	**30.** 6.	
31. (i) 24; (ii) 64;	(iii) 60.	**32.** 8.	**33.** 64.
34. 450.	**35.** 10.	**36.** 25.	**37.** 15.
38. 24.	**39.** (i) 20; (iii) 380.		**40.** 120.
41. (i) 40320.	**42.** 5040.	**43.** 720.	**44.** 720.
45. 120.	**46.** 8.	**47.** 336.	**48.** $2^3, 2^4, 2^5, 2n$.
49. 1920.	**50.** 6.		

Number of signals using five flags

Proceeding as earlier number of signals using five flags

$$= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720 \text{ signals}$$

Number of signals using six flags

Proceeding as earlier, number of signals using six flags

$$= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720 \text{ signals}$$

Total number of signals = (120 + 360 + 720 + 720) signals = 1920 signals

FACTORIAL NOTATION

We often come across products of the form $1 \cdot 2$, $1 \cdot 2 \cdot 3$, $1 \cdot 2 \cdot 3 \cdot 4$, ..., and so on.

Instead of writing all the factors of such a product in full, it is convenient to use a special notation. We write

$$1! = 1,$$
$$2! = 1 \cdot 2,$$
$$3! = 1 \cdot 2 \cdot 3,$$
$$\dots\dots\dots\dots$$
$$n! = 1 \cdot 2 \cdot 3 \dots, n.$$

"$n!$" denotes the product of the first n natural numbers. We read '$n!$' as 'n factorial'. $n!$ is also written as '$\underline{|n}$' and read as 'factorial n'. It is easy to see that

$$1! = 1,\ 2! = 1 \cdot 2 = 2,\ 3! = 1 \cdot 2 \cdot 3 = 6,\ 4! = 1 \cdot 2 \cdot 3 \cdot 4. = 24, \text{ and so on.}$$

IMPORTANT

- We know that

$$n! = n(n-1)(n-2)(n-3) \dots 3 \cdot 2 \cdot 1$$
$$= n(n-1)! = n(n-1)(n-2)!$$
$$= n(n-1)(n-2)(n-3)! \text{ and so on}$$

Thus, if $m,\ n \in N$ and $m > n$, then $m!$ can be expressed in terms of $n!$

For example $8! = 8 \cdot 7 \cdot 6!$

$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5!$

Also $m! = n!$ if and only if $m = n$.

- Putting $n = 1$ in $n! = n(n-1)!$, we have

$$1! = 1 \cdot 0!$$
$$\therefore \quad 0! = 1$$

CAUTION

- The factorial is defined only for whole numbers

We do not define the factorial of proper fractions.

Example 1. *Evaluate:*

(i) $\dfrac{30!}{28!}$; (ii) $\dfrac{9!}{5!\,3!}$; (iii) $\dfrac{12!-10!}{9!}$; (iv) $\dfrac{1}{3!}+\dfrac{1}{4!}+\dfrac{1}{5!}$.

(ii) 33! contains the following other numbers which contain atleast one 2 as their factor:

6, 10, 12, 14, 18, 20, 22, 24, 26, 28, 30

The product of '2' present in these numbers

$$= 2 \times 2 \times 2^2 \times 2 \times 2 \times 2^2 \times 2 \times 2^3 \times 2 \times 2^2 \times 2 = 2^{16} \qquad ...(2)$$

Combining (1) and (2), we get

$$2^{15} \times 2^{16} = 2^{31}$$

Hence 31 is the largest integer n such that 33! is divisible by 2^n.

EXERCISE 7.2

1. Find the values of :

(i) $\frac{11!}{8!}$; (ii) $\frac{6!}{4!2!}$; (iii) $\frac{(n+3)!}{n!}$; (iv) $\frac{8!-7!}{6!}$; (v) $\frac{1}{8!}+\frac{1}{9!}+\frac{1}{10!}$;

(vi) $6! \times (1 \cdot 3 \cdot 5 \ldots 11) \cdot 2^6$.

2. Evaluate $(n - r)!$, when

(i) $n = 6, r = 2$; (ii) $n = 10, r = 5$.

3. Evaluate $\frac{n!}{(n-r)!}$, when

(i) $n = 10, r = 4$; (ii) $n = 12, r = 3$; (iii) $r = 3$.

4. Compute $\frac{n!}{r!(n-r)!}$, when

(i) $n = 6, r = 2$; (ii) $n = 7, r = 4$; (iii) $n = 22, r = 2$.

5. Find the L.C.M. of the following:

(i) 5!, 6!, 7!; (ii) $(x - 1)!, x!, (x + 1)!$; (iii) 2 !, 4 !, 6!.

6. Find x, if

(i) $\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}=\frac{x}{7!}$; (ii) $\frac{1}{10!}+\frac{x}{2\cdot(11!)}=\frac{252}{12!}$.

7. If $\frac{(n+1)}{4!(n-1)!}$ and $\frac{(n+1)!}{6!(n-3)!}$ are in the ratio 3 : 2, find the value of n.

8. Convert into factorials:

(i) $5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12$; (ii) $4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14$.

9. Prove that: $\frac{(2n+1)!}{n!} = 1 \cdot 3 \cdot 5 \ldots (2n - 1)(2n + 1) \cdot 2^n$.

10. Prove that: $n!\,(n + z) = n! + (n + 1)!$.

11. If $(n + z)! = 2550(n)!$, find n.

12. If $(n + z)! = 60[(n - 1)!]$, find n.

13. If $(n + 1)! = 12[(n - 1)!]$, find n.

14. Show that $55! + 1$ is not divisible by any number between 2 and 55.

15. Prove that 29! is divisible by 2^{12}. What is the largest integer n such that 29! is divisible by 2^n?

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Example 4. *Find n if*

(i) ${}^nP_4 = 18 \cdot {}^{n-1}P_2$; (ii) ${}^nP_5 : {}^{n-1}P_4 = 6 : 1$.

Solution: (i) ${}^nP_4 = 18 \cdot {}^{n-1}P_2 \Rightarrow \dfrac{n!}{(n-4)!} = 18 \cdot \dfrac{(n-1)!}{(n-1-2)!}$

$$\Rightarrow \frac{n!}{(n-4)!} = 18 \cdot \frac{(n-1)!}{(n-3)!}$$

$$\Rightarrow \frac{n(n-1)!}{(n-4)!} = 18 \cdot \frac{(n-1)!}{(n-3)(n-4)!}$$

$\therefore$ $$n = \frac{18}{n-3}, \text{ i.e., } n(n-3) = 18$$

$$\Rightarrow n^2 - 3n - 18 = 0$$

$$\Rightarrow (n-6)(n+3) = 0 \Rightarrow n = 6, -3$$

But n cannot be negative.

$\therefore$ $n = 6$

(ii) $$\frac{{}^nP_5}{{}^{n-1}P_4} = \frac{6}{1} \Rightarrow \frac{n(n-1)(n-2)(n-3)(n-4)}{(n-1)(n-2)(n-3)(n-4)} = \frac{6}{1} \Rightarrow n = 6$$

Example 5. *If $P(56, r+6) : P(54, r+3) = 30800 : 1$, find r.*

Solution: $P(56, r+6) : P(54, r+3) = 30800 : 1$

$$\Rightarrow \frac{56!}{(50-r)!} : \frac{54!}{(51-r)!} = 30800 : 1 \Rightarrow \frac{54!\cdot 55.56}{(50-r)!} : \frac{54!}{(50-r)!\cdot(51-r)} = 30800 : 1$$

$$\Rightarrow \frac{3080}{1} : \frac{1}{51-r} = 30800 : 1 \Rightarrow 3080(51-r) = \frac{30800}{1}$$

$$\Rightarrow 51 - r = 10 \Rightarrow r = 51 - 10 \quad \therefore \quad r = 41$$

Example 6. *Find the value(s) of n such that:*

$$30P(n, 6) = P(n+2, 7);\ n \geq 6$$

Solution: $30P(n, 6) = P(n+2, 7)$

$$\Rightarrow \frac{30\cdot(n!)}{(n-6)!} = \frac{(n+2)!}{(n-5)!} \Rightarrow \frac{30\cdot(n!)}{(n-6)!} = \frac{(n+2)(n+1)\cdot(n!)}{(n-5)\cdot[(n-6)!]}$$

$$\Rightarrow 30 = \frac{n^2+3n+2}{n-5} \Rightarrow n^2 + 3n + 2 = 30n - 150$$

$$\Rightarrow n^2 - 27n + 152 = 0 \Rightarrow (n-8)(n-19) = 0$$

$$\Rightarrow n - 8 = 0 \text{ or } n - 19 = 0 \Rightarrow n = 8 \text{ or } n = 19$$

Hence the required values of n are $n = 8$ and $n = 19$.

Example 7. *Prove that:*

$$P(10, 3) = P(9, 3) + 3P(9, 2)$$

Solution: L.H.S. $= P(10, 3) = \dfrac{10!}{(10-3)!} = \dfrac{10!}{7!} = \dfrac{10\cdot 9\cdot 8\cdot 7!}{7!} = 720$

R.H.S. $= P(9, 3) + 3P(9, 2) = \dfrac{9!}{(9-3)!} + \dfrac{3.9!}{(9-2)!} = \dfrac{9!}{6!} + \dfrac{3\cdot 9!}{7!}$

$$= \frac{9\cdot 8\cdot 7\cdot 6!}{6!} + \frac{3\cdot 9\cdot 8\cdot 7!}{7!} = 504 + 216 = 720$$

Solution: Total no. of candidates = 5 + 4 = 9

In the row of 9 positions, the even places are 2nd, 4th, 6th and 8th.

Now number of even places = 4

No. of women to occupy the even places = 4

∴ Even places can be filled = $P(4, 4)$ ways

No. of men = 5

∴ The remaining 5 places can be filled by 5 men = $P(5, 5)$ ways.

By the fundamental principle of counting;

The required number of seating arrangements = $P(4, 4) \times P(5, 5)$

$= 4! \times 5! = 24 \times 120 = 2880$

Examples 5. *How many signals can be given with 5 flags of different colours.*

Solution: Since any number of flags can be used to give a signal, hence there are following possibilities:

(i) number of ways when only one flag is used = ${}^5P_1 = 5$

(ii) number of ways when any two flags are used = ${}^5P_2 = 20$

(iii) number of ways when any three flags are used = ${}^5P_3 = 60$

(iv) number of ways when any four flags are used = ${}^5P_4 = 120$

(v) number of ways when all five flags are used = ${}^5P_5 = 120$.

∴ The total number of signals = 5 + 20 + 60 + 120 + 120 = 325.

Example 6. *Four books, one each in Chemistry, Physics, Biology and Mathematics are to be arranged in a shelf. In how many ways can this be done?*

Solution: 4 different books can be arranged among themselves, in a shelf, in $P(4, 4)$ $= 4 \times 3 \times 2 \times 1 = 24$ ways.

Example 7. *There are three different rings to be worn in four fingers with at most one in each finger. In how many ways can this be done?*

Solution: Wearing 3 different rings in four fingers with at most one in each finger is equivalent to arranging 3 different objects in 4 places.

This can be done in $P(4, 3) = 4 \times 3 \times 2 = 24$ ways.

Example 8. *In an examination hall, there are four rows of chairs. Each row has 8 chairs one behind the other. There are two classes sitting for the examination with 16 students in each class. It is desired that in each row all students belong to the same class and that no two adjacent rows are allotted to the same class. In how many ways can these 32 students be seated?*

Solution: There are four rows of chairs (say I, II, III, IV) consisting of 8 chairs each. It is desired that in each row, all students belong to the same class and no two adjacent rows are allotted to same class. Therefore, one class can be seated in either I and III or in II and IV i.e., in 2 ways.

Now, 16 students of this class can be arranged in 16 chairs in ${}^{16}P_{16} = 16!$ ways.

16 students of the other class can be arranged in remaining 16 chairs in ${}^{16}P_{16} = 16!$ ways.

Total number of ways = $2 \times 16! \times 16!$

Example 9. *Find the number of different 8 letter words formed from the letters of the word EQUATION, if each word is to start with a vowel.*

Solution: Total number of candidates = 8

5 different subjects candidates can be seated in $P(5, 5) = 5!$ ways

In between 5 candidates there are 6 places for 8 mathematics candidates.

$\therefore$ The mathematics candidates can be seated in $P(6, 3)$ ways

$\therefore$ By the fundamental principle of counting:

Required number of ways $= 5! \times P(6, 3) = 120 \times \frac{6!}{3!} = 120 \times 6 \cdot 5 \cdot 4 = 14400$

Example 21. *How many different words can be formed of the letters of the word 'MALENKOV' so that*

(i) *no two vowels are together;*

(ii) *the relative position of the vowels and consonants remains unaltered?*

Solution: (i) No. of letters in the word 'MALENKOV' = 8

No. of vowels in the word 'MALENKOV' = 3 (i.e. A, E, O)

$\because$ No two vowels are together, so first we arrange the consonants

$$\times C \times C \times C \times C \times C \times$$

The 5-consonants can be arranged in $P(5, 5)$ ways. The vowels A, E, O can be arranged in the 6 '×' marked places in ways = $P(6, 3)$

$\therefore$ Total number of different words formed $= P(5, 5) \times P(6, 3) = 5! \times \frac{6!}{3!}$

$= 120 \times 6 \times 5 \times 4 = 14400$

(ii) The relative positions of the vowels and consonants remain unaltered which means a vowel can take the place of the other vowel and a consonant that of the other consonant.

Now, 3 vowels can be arranged in $P(3, 3)$ ways, and 5 consonants can be arranged $= P(5, 5)$ ways.

$\therefore$ Required number of words $= P(3, 3) \times P(5, 5) = 3! \times 5! = 6 \times 120 = 720$

Example 22. *Find how many words can be formed out of the letters of the word 'ORIENTAL' so that vowels always occupy the odd places.*

Solution: The vowels in the word 'ORIENTAL' are:

O, I, E and A.

Total no. of letters in the word 'ORIENTAL' = 8

No. of vowels = 4

O	*R*	*I*	*E*	*N*	*T*	*A*	*L*
1	2	3	4	5	6	7	8
×		×		×		×	

Vowels occupy odd places, i.e., 1, 3, 5 and 7.

No. of odd places = 4

$\therefore$ 4 vowels can be arranged in 4 '×' marked places $= P(4, 4)$ ways $= 4!$ ways

No. of consonants = 4

$\therefore$ 4 consonants can be arranged in 4 places $= P(4, 4)$ ways $= 4!$ ways

$\therefore$ Required no. of words $= 4! \times 4! = 24 \times 24 = 576$

Example 23. *In how many ways can the letters of the word PERMUTATIONS be arranged if the*

(i) *Words start with P and end with S?*

(ii) *Vowels are together?*

(iii) *There are always 4 letters between P and S?*

Example 34. *When a group photograph is taken, all the seven teachers should be in the first row and all the twenty students should be in the second row. If the two corners of the second row are reserved for the two tallest students, interchangeable only between them, and if the middle seat of the front row is reserved for the principal, how many arrangements are possible.?*

Solution: Since the middle seat of the first row is reserved for the principal, the remaining 6 teachers can be arranged in 6! ways.

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720 \text{ ways}$$

Again there are twenty students. The two corner seats are to be used by two tall students. The two tall students can be arranged in two ways.

The remaining 18 students can be arranged in 18! ways.

Total number of arrangements for the second row = 2 × 18! ways

$\therefore$ Required number of arrangements = 2 × 18! × 720 = 1440 × (18!)

Example 35. *The letters of the word "STRANGE" be arranged so that the vowels may appear in the odd places.*

Solution: There are 5 consonants and 2 vowels in the word STRANGE. Out of 7 places for the 7 letters, 4 places are odd and 3 places are even.

2 vowels can be arranged in 4 odd places in $P(4, 2)$ ways = 12 ways and then 5 consonants can be arranged in the remaining 5 places in $P(5, 5)$ ways.

$$5 \times 4 \times 3 \times 2 \times 1 = 120 \text{ ways}$$

Hence the required number of ways are:

$$P(4, 2) \times P(5, 5) = 12 \times 120 = 1440$$

Example 36. *Three married couples are to be seated in a row having six seats in a cinema hall. If the spouses are to be seated next to each other, in how many ways can they be seated? Find also the number of ways of their seating if all the ladies sit together.*

Solution: Let A, B and C are three couples. These couples can be seated in a row in $P(3, 3) = 3!$ ways. A, B and C can sit in $P(2, 2)$ ways each.

Hence the total number of permutations, is:

$$P(3,3) \times P(2,2) \times P(2,2) \times P(2,2) = 3!\,2!\,2!\,2!$$

$$6 \times 2 \times 2 \times 2 = 48 \text{ ways}$$

Second part:

Three ladies can sit in $P(3, 3) = 3!$ ways

Three men can sit in $P(3, 3) = 3!$ ways

Mutual arrangement of men and women = $P(2, 2) = 2$

If p is the number of ways, then

$$p = 3!\,3!\,2! \times 2 = 6 \times 6 \times 2 \times 2 = 144$$

(ii) To form even numbers with the digits 1, 2, 3, 4, 5; the unit place must be either 2 or 4.

Of the 1-digit numbers, only 2 are even (2 and 4).

Of the 2-digit numbers, even numbers = $2 \times {}^4P_1 = 2 \times 4 = 8$.

Of the 3-digit numbers, even numbers = $2 \times {}^4P_2 = 2 \times 4 \times 3 = 24$.

Of the four-digit numbers, even numbers = $2 \times {}^4P_3 = 2 \times 4 \times 3 \times 2 = 48$

Of the five-digit numbers, even numbers = $2 \times {}^4P_4 = 2 \times 4! = 2 \times 4 \times 3 \times 2 \times 1 = 48$

$\therefore$ Total number of even numbers = $2 + 8 + 24 + 48 + 48 = 130$.

27. The Word EQUATION consists of 5 vowels and 3 consonants. 5 vowels can be arranged in $5! = 120$ ways. 3 consonants can be arranged in $3! = 6$ ways.

The two block of vowels and consonants can be arranged in $2! = 2$ ways

$\therefore$ The number of words which can be formed with letters of the word EQUATION so that vowels and consonants occur together = $120 \times 6 \times 2 = 1440$.

28. There are 5 different letters in the word DELHI.

They are to occupy 5 places out of which 2 are even namely 2nd and 5th.

The two vowels namely E and I can be placed in the two even places (2nd and 4th) in 2! ways. Again the remaining three places (1st, 3rd, 5th) can be filled with the remaining three letters (D, L, H) in 3! ways.

$\therefore$ Required number of arrangements = $2! \times 3! = 2 \times 6 = 12$.

35. (i) Taking 3 sisters as 1 unit, we have to arrange 4 + 1 (= 5) different units taking 5 at a time in a row. This can be done in $P(5, 5)$ ways = $5 \times 4 \times 3 \times 2 \times 1 = 120$ ways.

In each of 120 ways, 3 sisters can be arranged themselves in $P(3, 3)$ ways

$$3 \times 2 \times 1 = 6 \text{ ways}$$

$\therefore$ The required no. of ways = $120 \times 6 = 720$.

(ii) 4 brothers can be arranged in $P(4, 4)$ ways = $4 \times 3 \times 2 \times 1 = 24$ ways.

In between and at the extremities of 4 brothers, there are 5 places in which 3 sisters can be arranged in $P(5, 3)$ ways = $5 \times 4 \times 3 = 60$ ways.

$\therefore$ Required number of arrangements = $24 \times 60 = 1440$.

36. $A = \{2, 3, 5, 7, 11, 13, 17, 19\ 23, 29\}$. A rational number is made by taking any two in any order.

$\therefore$ The required number of rational numbers = ${}^{10}P_2 + 1$ (including).

37. Here number of flags $n = 6$

Number of flags used $r = 6$

If p is the number of signals generated, then

$$p = P(6, 6) = \frac{6!}{(6-6)!} = \frac{6!}{0!}$$

$$p = 6! \qquad [\text{Q } 0! = 1]$$

$$= 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720.$$

38. Number of items in column $A = 6$ and number of items in colmn $B = 6$

If p is the number of answers, then

$$p = P(6, 6) = \frac{6!}{(6-6)!} \qquad \left[\because P(n,r) = \frac{n!}{(n-r)!}\right]$$

$$= \frac{6!}{0!} = 6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

(ii) Those words which begin with B and end at T, have only four places to be filled out of the remaining four letters, i.e., H, A, R, A out of which 2 are alike

$\therefore$ Number of such words $= \frac{4!}{2!} = 12$

Example 8. *How many arrangements can be made of the letters of the word 'ARRANGEMENT'. In how many of these the vowels occur together.*

Solution: The given word consists of 11 letters out of which A occurs 2 times, R occurs 2 times, N occurs 2 times and E occurs 2 times and remaining three are different.

$\therefore$ Number of arrangements $= \frac{11!}{2!2!2!2!} = 2491800$

Now, there are 4 vowels in the given word (A, A, E, E). Let us treat these 4 vowels as one letter (AAEE). Now, there are 8 letters [(AAEE), R, R, N, G, M, N, T] out of which R and N occur 2 times each.

$\therefore$ Number of arrangements $= \frac{8!}{2!2!} = 10080$

Corresponding to each such arrangement, the four vowels A, A, E, E, out of which A and E occur 2 times each, can be arranged amongest themselves in $\frac{4!}{2!2!} = 6$ ways.

Hence the total number of arrangements $= 10080 \times 6 = 60480$

Example 9. *If the different permutations of the word EXAMINATION are listed as in a dictionary, how many items are there in this list before the first word starting with E?*

Solution: Starting with A and arranging the other ten letters A, E, I, I, M, N, N, O, T, X (not all distinct, I occurs twice, N occurs twice), there are

$$\frac{10!}{2!2!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 2} = 907200 \text{ words}$$

The number of items in the list before the first word starting with E is 907200.

Example 10 *How many 5 digit even numbers can be formed using the digits 1, 2, 5, 5, 4.*

Solution: The 5 digit even numbers can be formed out of 1, 2, 5, 5, 4 by using either 2 or 4 in the uni's place. This can be done in 2 ways.

Corresponding to each such arrangement, the remaining 4 places can be filled up by any of the remaining four digits in $\frac{4!}{2!} = 12$ ways. [$\because$ 50 occurs twice]

Hence the total number of words $= 2 \times 12 = 24$

Example 11. *How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3.*

Solution: A number greater than a million has 7 places, and thus all the 7 given digits are to be used.

But 2 is repeated twice and 3 is repeated thrice.

$\therefore$ Total number of ways of arranging these 7 digits amongst themselves

$$= \frac{7!}{2! \cdot 3!} = 420$$

Solution: In going from A to B, no matter which path the person chooses, he must walk 4 right and 5 ups total 9. Therefore, here, to get different paths, person has to get arrangement of 9 things in which 4 (right) are of one type and five (ups) are of other kind.

$$\therefore \quad \text{Number of different paths} = \frac{9!}{4!5!} = \frac{1.2.3.4.5.6.7.8.9}{1.2.3.4 \times 1.2.3.4.5} = 2 \times 7 \times 9 = 126$$

Example 22. *A biologist studying the genetic code is interested to know the number of possible arrangements of 12 molecules in a chain. The chain contains 4 different molecules represented by the initials A (for Adenine), C (for Cytosine), G (for Guanine) and T (for Thymine) and 3 molecules of each kind. How many different arrangements are possible in all?*

Solution: The number of required ways

$$= \frac{12}{3!3!3!3!} = \frac{12 \times 11 \times 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3!}{3 \times 2 \times 1 \times 3 \times 2 \times 1 \times 3 \times 2 \times 1 \times 3!} = 369600$$

EXERCISE 7.5

1. In how many ways can 4 prizes be given to 3 boys when a boy is eligible for all the prizes.
2. A letter lock consists of 4 rings, each marked with 6 different letters; find in how many ways it is possible to make an unsuccessful attempt to open the lock.
3. In how many ways can 5 apples be distributed among 4 boys, there being no restriction to the number of apples each boy may get.
4. A boy has 3 pockets. In how many ways can he put 6 marbles in his pockets?
5. How many four digit numbers can be formed with the digit 1, 2, 3, 4, 5, 6 when a digit may be repeated any number of times in any arrangement?
6. How many numbers less than 100 can be formed with the digits 1, 3, 5, 7, 9 when the digits may be repeated?
7. Find the number of permutations that can be made out of the letters of the words:
   (i) INDIA; (ii) EXAMINATION; (iii) INDEPENDENCE;
   (iv) ASSASSINATION; (v) ACCOMODATION.
8. How many 3 digits numbers can be formed by using the digits 0, 1, 3, 5, 7 when the digits may be repeated any number of times.
9. Find the number of arrangements that can be made out of the letters of the word PERMUTATION. In how many of these 5 vowels occur together.
10. There are three blue balls, four red balls and five green balls. In how many ways can they be arranged in a row?
11. Find the number of arrangements that can be made out of the letters of the word MATHEMATICS. In how many of these vowels occur together.
12. How many 7 digit numbers can be formed using the digits 1, 2, 0, 2, 4, 2 and 4.
13. How many arrangements can be made with the letters of the word 'SERIES'? How many of these begin and end with 'S'?
14. There are 3 prizes to be distributed among 5 students. In how many ways can it be done when
    (i) no student gets more than one prize;
    (ii) there is no restriction as to the number of prizes any student gets (i.e., a student may get any number of prizes);
    (iii) no student gets all the prizes?

out of these in 4 ways, namely, ABC, ABD, ACD and BCD. We say that there are four combinations of 4 letters taken 3 at time. In symbols, we write ${}^4C_3 = 4$ or $C(4, 3) = 4$.

Each of the four combinations written above gives rise to 3! permutations. For example A, B, C can be arranged as

$$ABC,\ ACB,\ BCA,\ BAC,\ CAB,\ CBA$$

ABC, ACB, BCA, BAC, CAB, CBA represent different permutations, but all of them represent the same combination. From the above discussion, we find that ${}^4P_3 = {}^4C_3 \times (3!)$.

We shall see that a similar relation holds between ${}^nP_r$ and ${}^nC_r$. In fact, we shall prove that

$${}^nP_r = {}^nC_r \times (r!)$$

To Find the Number of Combinations of n Dissimilar Objects Taken r (≥ 1) at a Time:

Let the required number of combinations be denoted by ${}^nC_r$.

Each of these combinations is a collection of r dissimilar objects which can be arranged among themselves in $r!$ ways. Hence each combination gives rise to $r!$ permutations. Hence ${}^nC_r$ combinations will give rise to ${}^nC_r \times r!$ permutations. But the number of permutations of n things taken r at a time is ${}^nP_r$.

$$\therefore \qquad {}^nC_r \times r! = {}^nP_r,$$

$$\text{or} \qquad {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n(n-1)\ldots(n-r+1)}{r!} \qquad \ldots(1)$$

$$\text{or} \qquad {}^nC_r = \frac{{}^nP_r}{r!} = \frac{n!}{(n-r)!\,r!} \qquad \ldots(2)$$

$$\text{Hence} \qquad {}^nC_r = \frac{n(n-1)\ldots(n-r+1)}{r!} = \frac{n!}{(n-r)!\,r!}$$

Note: It is convenient to use (1) when a numerical result is required and (2) when it is sufficient to leave the result in the factorial notation.

Some Useful Results on Combinations

1. *For $0 \le r \le n$, prove that ${}^nC_r = {}^nC_{n-r}$*

Proof. $${}^nC_{n-r} = \frac{n!}{(n-r)!\,[n-(n-r)]!} \qquad \left[\because {}^nC_r = \frac{n!}{r!\,(n-r)!}\right]$$

$$= \frac{n!}{(n-r)!\,r!} = {}^nC_r$$

Hence $\quad {}^nC_r = {}^nC_{n-r}$.

Remark. If ${}^nC_p = {}^nC_q \Rightarrow {}^nC_p = {}^nC_q = {}^nC_{n-q}$ $\qquad [\because {}^nC_r = {}^nC_{n-r}]$

$\Rightarrow$ either $p = q$ or $p = n - q$

$\Rightarrow p = q$ or $p + q = n$

Thus, $\quad {}^nC_p = {}^nC_q \Rightarrow p = q$ or $p + q = n$

2. *If n and r are natural numbers such that $1 \le r \le n$, then ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$*

Proof. L.H.S. $= {}^nC_r + {}^nC_{r-1} = \dfrac{n!}{r!\,(n-r)!} + \dfrac{n!}{(r-1)!\,(n-r+1)!}$

$$= \frac{n!}{r(r-1)!\,(n-r)!} + \frac{n!}{(r-1)!\,(n-r+1)\,(n-r)!}$$

Example 10. *Prove that:* $\sum_{r=1}^{5} C(5, r) = 31.$

Solution: $\sum_{r=1}^{5} C(5, r) = C(5, 1) + C(5, 2) + C(5, 3) + C(5, 4) + C(5, 5)$

$$= \frac{5!}{1!4!} + \frac{5!}{2!3!} + \frac{5!}{3!2!} + \frac{5!}{4!1!} + \frac{5!}{5!0!} = \frac{5}{1} + \frac{5\cdot4}{2} + \frac{5\cdot4}{2} + \frac{5}{1} + 1$$

$$= 5 + 10 + 10 + 5 + 1 = 31$$

Example 11. *If* $C(n, r) : C(n, r+1) = 1 : 2$ *and* $C(n, r+1) : C(n, r+2) = 2 : 3$, *determine the values of n and r.*

Solution: $C(n, r) : C(n, r + 1) = 1 : 2$

$$\Rightarrow \quad \frac{n!}{r!(n-r)!} : \frac{n!}{(r+1)!(n-r-1)!} = 1 : 2$$

$$\text{or} \quad \frac{n!}{r!(n-r)(n-r-1)!} \times \frac{(r+1)r!(n-r-1)!}{n!} = \frac{1}{2}$$

$$\text{or} \quad \frac{r+1}{n-r} = \frac{1}{2} \quad \text{or} \quad n - 3r - 2 = 0 \qquad \text{...(1)}$$

Also $C(n, r + 1) : C(n, r + 2) = 2 : 3$

$$\Rightarrow \quad \frac{n!}{(r+1)!(n-r-1)!} : \frac{n!}{(r+2)!(n-r-2)!} = 2 : 3$$

$$\text{or} \quad \frac{n!}{(r+1)!(n-r-1)(n-r-2)!} \times \frac{(r+2)(r+1)!(n-r-2)!}{n!} = \frac{2}{3}$$

$$\text{or} \quad \frac{r+2}{n-r-1} = \frac{2}{3} \quad \text{or} \quad 2n - 5r - 8 = 0 \qquad \text{...(2)}$$

Solving (1) and (2), $n = 14$, $r = 4$.

EXERCISE 7.6

1. Evaluate the following:
   (i) $C(9, 3)$; (ii) $C(10, 8)$; (iii) ${}^{51}C_{49}$; (iv) $C(9, 5) + C(9, 6)$;
   (v) $C(11, 7) - C(10, 6)$; (vi) $C(7, 4) + C(7, 5) + C(8, 6)$.
2. If $C(n, 11) = C(n, 4)$, then find $C(n, 3)$.
3. Verify that $C(9, 6) = 3C(8, 2)$.
4. If $C(2n, 3) : C(n, 2) = 12 : 1$, then find n.
5. If ${}^{15}C_{3r} = {}^{15}C_{r+3}$, find r.
6. If $C(n, 8) = C(n, 6)$, find $C(n, 2)$.
7. If ${}^{4}P_2 = n\,[{}^{4}C_2]$, then find n.
8. Show that $C(47, 4) + \sum_{j=1}^{5} C(52 - j, 3) = C(52, 4)$.

$\therefore$ The total number of ways = ${}^6C_3 \cdot {}^3C_2 = 20 \times 3 = 60.$

(b) For including atleast one girl, the following cases arise:

(i) 1 girl + 4 boys, i.e., in ${}^3C_1 \cdot {}^6C_4 = 3 \times 15 = 45$ ways

(ii) 2 girls + 3 boys, i.e., in ${}^3C_2 \cdot {}^6C_3 = 3 \times 20 = 60$ wyas

(iii) 3 girls + 2 boys, i.e., in ${}^3C_3 \cdot {}^6C_2 = 1 \times 15 = 15$ ways

$\therefore$ Total number of committees = 45 + 60 + 15 = 120

Example 6. *A man has 7 friends. In how many ways can he invite one or more of them to a party.*

Solution: A man may invite one of them, two of them, ..., all of them and this can be done in ${}^7C_1$, ${}^7C_2$, ${}^7C_3$... ${}^7C_7$ ways.

$$\therefore \quad \text{Total number of ways} = {}^7C_1 + {}^7C_2 + {}^7C_3 + {}^7C_4 + {}^7C_5 + {}^7C_6 + {}^7C_7$$
$$= 7 + 21 + 35 + 21 + 7 + 1 = 127$$

Example 7. *From a class of* 12 *boys and* 10 *girls,* 10 *students are to be chosen for a competition, including atleast* 4 *boys and* 4 *girls. The two girls who won the prizes last year should be included. In how many ways can the selection be made?*

Solution: There are 12 boys and 10 girls in the class. We have to select 10 students for a competition including atleast 4 boys and 4 girls. Two girls who were last year's winner are to be included. Since two girls are already selected now we are left with 8 girls out of which atleast 2 girls are to be selected.

We can make selection in the following ways:

Choice	*Boys*	*Girls (+2 particular girls)*
I	4	4 + 2
II	5	3 + 2
III	6	2 + 2

First choice can be made in ${}^{12}C_4 \times {}^8C_4$

$$= \frac{12\times11\times10\times9}{4\times3\times2\times1}\times\frac{8\times7\times6\times5}{4\times3\times2\times1} = 34650 \text{ ways}$$

Second choice can be made in ${}^{12}C_5 \times {}^8C_3$

$$= \frac{12\times11\times10\times9\times8}{5\times4\times3\times2\times1}\times\frac{8\times7\times6}{3\times2\times1} = 44352 \text{ ways}$$

Third choice can be made in ${}^{12}C_6 \times {}^8C_2$

$$= \frac{12\times11\times10\times9\times8\times7}{6\times5\times4\times3\times2\times1}\times\frac{8\times7}{2\times1} = 25872 \text{ ways}$$

Hence, total number of possible selections

$$= 34650 + 44352 + 25872 = 104874$$

Example 8. *A question paper has two parts, part A and part B, each containing* 10 *questions. If the student has to choose* 8 *from part A and* 5 *from part B, in how many ways can he choose the question?*

Solution: The required number of ways = $C(10, 8) \cdot C(10, 5)$

$$= \frac{10!}{8!\,2!}\times\frac{10!}{5!\,5!} = \frac{10\times9}{2}\times\frac{10\times9\times8\times7\times6}{5\times4\times3\times2}$$

$$= 5 \times 9 \times 3 \times 2 \times 7 \times 6 = 11340$$

Example 9. *There are* 15 *points in a plane, no three of which are collinear except* 6 *of them which are all on a line. How many* (i) *st. lines;* (ii) *triangles can be formed by joining them?*

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Example 18. *Among 20 members of a cricket club, there are two wicket-keepers and 5 bowlers. In how many ways can eleven be chosen so as to include only one of the wicket-keepers and atleast three bowlers?*

Solution: No. of wicket-keepers = 2

No. of bowlers = 5

and No. of other players = 20 – 7 = 13.

11 players including 1 wicket-keeper and at least 3 bowlers can be selected out of 2 wicket-keepers, 5 bowlers and 13 other players by taking

(i) 1 wicket-keeper out of 2, 3 bowlers out of 5 and 7 other players out of 13, or

(ii) 1 wicket-keeper out of 2, 4 bowlers out of 5 and 6 other players out of 13 or

(iii) 1 wicket-keeper out of 2, 5 bowlers out of 5 and 5 other players out of 13

In case (i), the no. of ways = ${}^2C_1 \times {}^5C_3 \times {}^{13}C_7 = 2 \times 10 \times 1716 = 34320$

In case (ii), the no. of ways = ${}^2C_1 \times {}^5C_4 \times {}^{13}C_6 = 2 \times 5 \times 1716 = 17160$

In case (iii), the no. of ways = ${}^2C_1 \times {}^5C_4 \times {}^{13}C_6 = 2 \times 1 \times 1287 = 2574$

Hence the required no. of ways = 34320 + 17160 + 2574 = 54054

Example 19. *Out of 7 consonants and 4 vowels, how many words can be made each containing 3 consonants and 2 vowels?*

Solution: No. of ways of selecting 3 consonants from 7 consonants = $C(7, 3)$

No. of ways of selecting 2 vowels from 4 vowels = $C(4, 2)$

No. of ways of arranging 3 consonants and 2 vowels = 5!

$\therefore$ Required no. of words = $C(7, 3) \times C(4, 2) \times 5!$

$$= \frac{7!}{3!\,4!} \times \frac{4!}{2!\,2!} \times 120$$

$$= \frac{7 \times 6 \times 5}{6} \times \frac{4 \times 3}{2} \times 120$$

$$= 35 \times 6 \times 120 = 25200$$

Example 20. *How many words, each of 3 vowels and 2 consonants, can be formed from the letters of the word 'INVOLUTE'?*

Solution: No. of letters in the word = 8

No. of vowels in the word = 4

(I, O, U, E)

No. of consonants in the word = 4

(N, V, L, T)

Out of 4 vowels, we have to select 3.

Out of 4 consonants, we have to select 2.

Also we have to arrange 3 vowels and 2 consonants.

$\therefore$ Required no. of words = $C(4, 3) \cdot C(4, 2) \cdot 5! = 4 \times \dfrac{4 \times 3}{2} = 120 = 2880$

Example 21. *A committee of 12 is to be formed from 9 women and 8 men. In how many ways this can be done if atleast five women have to be included in a committee? In how many of these committees (i) the women are in majority and (ii) the men are in majority?*

Solution: The committee can be selected in the following way:

Men	7	6	5	4	3
Women	5	6	7	8	9

Example 31. *In an examination, Yamini has to select 4 questions from each part. There are 6, 7 and 8 questions in Part I, Part II and Part III respectively. What is the number of possible combinations in which she can choose the questions?*

Solutions: If p is the number of ways of selection, then

$$p = C(6, 4) \times C(7, 4) \times C(8, 4) = \frac{6!}{4!2!} \times \frac{7!}{4!3!} \times \frac{8!}{4!4!}$$

$$= \frac{6 \times 5 \times 4!}{2 \times 1 \times 4!} \times \frac{7 \times 6 \times 5 \times 4!}{3 \times 2 \times 1 \times 4!} \times \frac{8 \times 7 \times 6 \times 5 \times 4!}{4 \times 3 \times 2 \times 1 \times 4!}$$

$$= 15 \times 35 \times 70 = 36750 \text{ ways.}$$

Example 32. *We wish to select 6 persons from 8, but if the person A is chosen, then B must be chosen. In how many ways can the selection be made?*

Solution: If A and B both are not selected, then the number of permutations $= C(6, 6) = 1$.

If A and B both are selected, then we are to select 4 persons out of six persons. The number of ways in which this can be done is:

$$C(6, 4) = \frac{6!}{4!(6-2)!} = \frac{6 \times 5 \times 4!}{2 \times 1 \times 4!} = 15$$

Hence, total number of permutations = 1 + 15 = 16.

Example 33. *Determine the number of 5 cards combinations out of a deck of 52 cards if at least one of the 5 cards has to be a king?*

Solution: Combination of 5 cards may be in the following ways:

	King cards	Other cards
(i)	1	4
(ii)	2	3
(iii)	3	2
(iv)	4	1

If p_1, p_2, p_3 and p_4 are the number of permutations of (i), (ii), (iii) and (iv), then

$$p_1 + p_2 + p_3 + p_4 = C(4, 1) \times C(48, 4) + C(4, 2) \times C(48, 3) + C(4, 3) \times C(48, 2) + C(4, 4) \times C(48, 1)$$

$$= \frac{4!}{1!3!} \times \frac{48!}{4!44!} + \frac{4!}{2!2!} \times \frac{48!}{3!45!} + \frac{4!}{3!1!} \times \frac{48!}{2!46!} + \frac{4!}{4!0!} \times \frac{48!}{1!47!}$$

$$= \frac{4 \times 3!}{3!} \times \frac{48 \times 47 \times 46 \times 45 \times 44!}{4 \times 3 \times 2 \times 1 \times 44!} + \frac{4 \times 3 \times 2!}{2 \times 1 \times 2!} \times \frac{48 \times 47 \times 46 \times 45!}{3 \times 2 \times 1 \times 45!}$$

$$+ \frac{4 \times 3!}{3!1!} \times \frac{48 \times 47 \times 46!}{1 \cdot 2 \times 46!} + \frac{4!}{4!0!} \times \frac{48 \times 47!}{48!}$$

$$= 8 \times 47 \times 46 \times 45 + 6 \times 8 \times 47 \times 46 + 4 \times 24 \times 47 + 48$$

$$= 886656$$

EXERCISE 7.7

1. Find the number of combinations of 50 things taken 46 at a time.
2. In how many ways can a party of 4 be selected from 10 persons?

29. Number of friends = 7

The men can invite one friend, two friends ...., or seven friends.

$\therefore$ Required number of ways $= {}^7C_1 + {}^7C_2 + {}^7C_3 + {}^7C_4 + {}^7C_5 + {}^7C_6 + {}^7C_7$

$$= \frac{7}{1} + \frac{7\times6}{1\times2} + \frac{7\times6\times5}{1\times2\times3} + \frac{7\times6\times5}{1\times2\times3} + \frac{7\times6}{1\times2} + \frac{7}{1} + 1$$

$$= 7 + 21 + 35 + 35 + 21 + 7 + 1 = 127$$

30. First part. 3 ladies out of 8 can be selected in ${}^8C_3$ ways and 4 gentlemen out of 7 in ${}^7C_4$ ways.

Now, each way of selecting 3 ladies is associated with each way of selecting 4 gentlemen. Hence the required number of ways $= {}^8C_3 \times {}^7C_4 = 56 \times 35 = 1960$.

Second part. First we find the no. of committees of 3 ladies and 4 gentlemen in which both Mrs. X and Mr. Y are members. In this case, we can select 2 other ladies from the remaining 7 in ${}^7C_2$ ways and 3 other gentlemen from the remaining 6 in ${}^6C_3$ ways.

$\therefore$ The no. of ways in which both Mrs. X and Mr. Y are always included

$$= {}^7C_2 \times {}^8C_3 = 21 \times 20 = 420$$

Hence the required no. of committees in which Mrs. X and Mr. Y do not serve together

$$= 1960 - 420 = 1540$$

31. There are two different ways of forming the committee

(i) oldest may be included

(ii) oldest may be excluded

(i) If oldest is included, then youngest has to be excluded and we are to select 4 candidates out of 8. This can be done in ${}^8C_4$ ways.

$${}^8C_4 = \frac{8!}{4!4!} = \frac{8\times7\times6\times5}{4\times3\times2} = 70 \text{ ways}$$

(ii) If oldest is excluded, then we are to select 5 candidates from 9 which can be done in ${}^9C_5$ ways

$${}^9C_5 = \frac{9!}{5!4!} = \frac{9\times8\times7\times6}{4\times3\times2\times1} = 126 \text{ ways}$$

Hence the total number of ways in which committee can be formed

$$= 126 + 70 = 196$$

32. (i) ${}^{11}C_4$; (ii) ${}^{11}C_5$.

33. As no two lines are parallel and no three are concurrent, only two lines can meet at one point and every two lines meet at a point.

$\therefore$ They will intersect in ${}^{20}C_2$ points

$${}^{20}C_2 = \frac{20!}{18!(2!)} = \frac{20\times19\times18!}{18!\times2!} = \frac{20\times19}{2} = 190.$$

34. The plus signs can be arranged in only one way, because all are identical, as shown below:

	+		+		+		+		+		+		+	

A blank box in the above arrangement shows available space for the minus signs. Since there are 7 plus signs, the number of blank boxes is therefore 8. The five minus signs are now to be arranged in the 8 boxes so that no two of them are together. Now, 5 boxes out of 8 can be chosen in ${}^8C_5$ ways. Since all minus signs are identical, so 5 minus signs can be arranged in 5 chosen boxes in only one way. Hence, the number of possible arrangements $= 1 \times {}^8C_5 \times 1 = 56$.

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Solution: $(x + a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + {}^nC_3x^{n-3}a^3 + \cdots$

or $\quad (x + a)^n = ({}^nC_0x^n + {}^nC_2x^{n-2}a^2 + \cdots) + ({}^nC_1x^{n-1}a + {}^nC_3x^{n-3}a^3 + \cdots) \quad$ ...(1)

or $\quad (x + a)^n = P + Q \quad$ ...(2)

where $\quad P = {}^nC_0x^n + {}^nC_2x^{n-2}a^2 + \cdots$ and $Q = {}^nC_1x^{n-1}a + {}^nC_3x^{n-3}a^3 + \cdots$

Changing a to $-a$ in (1), we have

$$(x - a)^n = ({}^nC_0x^n + {}^nC_2x^{n-2}a^2 + \cdots) - ({}^nC_1x^{n-1}a + {}^nC_3x^{n-3}a^3 + \cdots)$$

or $\quad (x - a)^n = P - Q \quad$ ...(3)

(i) Multiplying (2) and (3),

$$(x + a)^n (x - a)^n = (P + Q)(P - Q)$$

or $\quad (x^2 - a^2)^n = P^2 - Q^2$

(ii) Squaring (2) and (3) and subtracting, we get

$$(x + a)^{2n} - (x - a)^{2n} = (P + Q)^2 - (P - Q)^2 = 4PQ$$

(iii) Squaring (2) and (3) and adding, we get

$$(x + a)^{2n} + (x - a)^{2n} = (P + Q)^2 + (P - Q)^2 = 2(P^2 + Q^2)$$

Example 6. *If x and y are distinct integers, prove that $x^n - y^n$ is divisible by $x - y$, whenever n is a positive integer.*

Solution: Since $x = (x - y) + y$, we have

$$x^n = [(x - y) + y]^n,$$

Expand R.H.S. by binomial theorem

$$= {}^nC_0(x - y)^n + {}^nC_1(x - y)^{n-1}y + {}^nC_2(x - y)^{n-2}y^2 + \cdots + {}^nC_{n-1}(x - y)y^{n-1} + {}^nC_ny^n$$

Transposing y^n to the L.H.S., we get

$$x^n - y^n = {}^nC_0(x - y)^n + {}^nC_1(x - y)^{n-1}y + \cdots + {}^nC_{n-1}(x - y)y^{n-1}$$
$$= (x - y)[{}^nC_0(x - y)^{n-1} + {}^nC_1(x - y)^{n-2}y + \cdots + {}^nC_{n-1}y^{n-1}]$$
$$= (x - y) \times \text{Some integer}$$

Hence $x^n - y^n$ is divisible by $x - y$ for all +ve integral n.

Example 7. *Prove that* $\sum_{r=0}^{n} 3^r C(n, r) = 4^n$.

Solution: $\sum_{r=0}^{n} 3^rC(n, r) = 3^0C(n, 0) + 3^1C(n, 1) + 3^2C(n, 2) + \cdots + 3^nC(n, n) = (1 + 3)^n = 4^n$

Example 8. *Write down the binomial expansion of $(1 + x)^{n+1}$, when $x = 8$. Deduce that $9^{n+1} - 8n - 9$ is divisible by 64, whenever n is a positive integer.*

Solution: $(1 + x)^{n+1} = (1 + 8)^{n+1} = 1 + C(n + 1, 1).\ 8 + C(n + 1, 2) \cdot 8^2 + C(n + 1, 3)\ 8^3 + \cdots + C(n + 1, n + 1).\ 8^{n+1}$

$$= 1 + (n + 1) \cdot 8 + C(n + 1, 2).\ 8^2 + C(n + 1, 3) \cdot 8^3 + \cdots + C(n + 1, n + 1) \cdot 8^{n+1}$$

$\Rightarrow \quad 9^{n+1} - 8n - 9 = 8^2[C(n + 1, 2) + 8\,C(n + 1, 3) + \cdots + 8^{n-1}]$

$$= 64 \times \text{A positive integer}$$

Therefore, $\quad 9^{n+1} - 8n - 9$ is divisible by 64.

20. By binomial theorem,

$$(1 + x)^n = \left[1 + nx + \frac{n(n-1)}{2} \cdot x^2 ... x^n\right]$$

or $$(1 + x)^n - 1 = nx + \frac{n(n-1)}{2}x^2 ... x^n$$

If $$x = n,\ (1 + n)^n - 1 = n^2 + \frac{n(n-1)}{2}n^2 ... n^n$$

$$(1 + n)^n - 1 = n^2\left[1 + \frac{n(n-1)}{2} ... n^{n-2}\right]$$

Put $$n = 100,\ (1 + 100)^{100} - 1 = (100)^2\left[1 + \frac{100(100-1)}{2} ... 100^{98}\right]$$

$$(101)^{100} - 1 = (100)^2\left[1 + \frac{100 \times 99}{2} ... 100^{98}\right]$$

Clearly $(101)^{100} - 1$ is divisible by $(100)^2 = 10000$.

GENERAL TERM IN A BINOMIAL EXPANSION

In the expansion of $(x + a)^n$, the $(r + 1)th$ term, i.e., T_{r+1} is called its *general term*. We find that

$$T_1 = {}^nC_0 x^n$$
$$T_2 = {}^nC_1 x^{n-1} a$$
$$T_3 = {}^nC_2 x^{n-2}.\ a^2 ...$$
$$...................................$$

$\therefore$ $$T_{r+1} = {}^nC_r \cdot x^{n-r} \cdot a^r$$

$\therefore$ General term $= {}^nC_r x^{n-r} a^r$.

IMPORTANT

The $(r + 1)th$ term from the end in the expansion of $(x + a)^n$ or $(1 + x)^n$ is equal to $(n - r + 1)th$ term from the beginning in the same expansion.
Since in the expansion of $(x + a)^n$ or $(1 + x)^n$, there are $(n + 1)$ terms, therefore, $(r + 1)th$ term from the end will have $(n + 1) - (r + 1) = n - r$ terms before it from the beginning. Hence $(r + 1)th$ term from the end $= (n - r + 1)th$ term from the beginning in the same expansion.

OR the $(r + 1)th$ term from the end in the expansion of $(x + a)^n = (r + 1)th$ term from the beginning in $(a + x)^n$.

Middle Term(s) in the Expansion of $(x + a)^n$, $n \in N$

If the power of the binomial, i.e., n is even, the number of terms is odd, i.e., $n + 1$ and, therefore, there is only one middle term. If n is odd, the number of terms is even and, therefore, there are two middle terms. Thus,

(i) *If n is even, there is one middle term,* i.e., $\left(\frac{n}{2} + 1\right)th$.

(ii) *If n is odd, there are two middle terms,* i.e., $\left(\frac{n+1}{2}\right)th$ *and* $\left(\frac{n+1}{2} + 1\right)th$.

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Solution: (i) The given expression is $\left(2x - \frac{1}{y}\right)^7$.

Here $n = 7$, which is odd. There are two mid terms, i.e., $T_{\frac{n+1}{2}}$ and $T_{\frac{n+1}{2}+1}$, i.e., T_4 and T_5.

Now, $$T_4 = {}^7C_3(2x)^4\left(-\frac{1}{y}\right)^3 = \frac{7.6.5}{3.2.1}\cdot 16\,x^4\left(-\frac{1}{y^3}\right) = -\frac{560\,x^4}{y^3}$$

and $$T_5 = {}^7C_4(2x)^3\left(-\frac{1}{y}\right)^4 = \frac{7.6.5.4}{4.3.2.1}\cdot 8x^3\frac{1}{y^4} = \frac{280\,x^3}{y^4}$$

(ii) Here $n = 2n$, which is even. There is only one mid term, i.e., $T_{\frac{2n}{2}+1} = T_{n+1}$.

Now, $$T_{n+1} = {}^{2n}C_n\,(x)^n\left(-\frac{1}{x}\right)^n = (-1)^n\frac{(2n)!}{(n!)^2}$$

Example 11. *Find the terms containing x^2, if any, in the expansion of* $\left(3x - \frac{1}{2x}\right)^8$.

Solution: The given expression is $\left(3x - \frac{1}{2x}\right)^8$.

Here $n = 8$, $x = 3x$ and $a = -\frac{1}{2x}$.

$$\therefore \quad T_{r+1} = {}^8C_r(3x)^{8-r}\left(-\frac{1}{2x}\right)^r = {}^8C_r3^{8-r}\,x^{8-r}\frac{(-1)^r}{2^r\cdot x^r}$$

$$= (-1)^r\frac{3^{8-r}}{2^r}\,{}^8C_rx^{8-2r} \qquad ...(1)$$

T_{r+1} will contain x^2 if $8 - 2r = 2$, i.e., if $r = 3$. Putting $r = 3$ in (1), we get

$$T_4 = (-1)^3\frac{3^{8-3}}{2^3}\;{}^8C_3x^{8-6} = -\frac{1.3^5}{2^3}\cdot\frac{8.7.6}{1.2.3}\,x^2 = -1701x^2$$

The term containing x^2 is T_4 and it is $-1701x^2$.

Example 12. *Which term contains x^8 in the expansion of* $\left(x^2 - \frac{1}{x}\right)^{10}$ *? Also find its coefficient.*

Solution: The given expression is $\left(x^2 - \frac{1}{x}\right)^{10}$.

Here $n = 10$, $x = x^2$ and $a = -\frac{1}{x}$.

$$T_{r+1} = {}^{10}C_r(x^2)^{10-r}\left(\frac{-1}{x}\right)^r = {}^{10}C_rx^{20-2r}\frac{(-1)^r}{x^r}$$

or $$T_{r+1} = (-1)^r\,{}^{10}C_rx^{20-3r} \qquad ...(1)$$

To find the term containing x^8, put $20 - 3r = 8$ or $-3r = -12$, i.e., $r = 4$. Putting $r = 4$ in (1), we get

$$T_5 = (-1)^4\,{}^{10}C_4x^8 = \frac{10\times9\times8\times7}{4\times3\times2\times1}\,x^8 = 210x^8$$

The 5*th* term contains x^8 and its coefficient is 210.

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or $\quad 6^{\frac{n}{4}-2}=6^{\frac{1}{2}} \quad \therefore \ \frac{n}{4}-2=\frac{1}{2} \quad \therefore \ \frac{n}{4}=\frac{5}{2}$ or $n=\frac{5}{2}\times 4=10$

EXERCISE 8.2

LEVEL OF DIFFICULTY A

1. Find:

(i) 4th term of $\left(\frac{x}{a}-\frac{a}{x}\right)^{10}$; (ii) 7th term of $\left(\frac{x^2}{2}-\frac{2}{x^2}\right)^9$;

(iii) 10th term of $\left(2x^2+\frac{1}{x}\right)^{12}$; (iv) General term of $(x^2-yx)^{12}, x\neq 0$;

(v) 8th term of $(x^{3/2}\,y^{1/2}-x^{1/2}y^{3/2})^{10}$;

(vi) 11th term of $\left(4x-\frac{1}{2\sqrt{x}}\right)^{15}$.

2. Find:

(i) 5th term from the end in $\left(\frac{x^3}{2}-\frac{2}{x^2}\right)^6$;

(ii) 4th term from the end in $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$;

(iii) 4th term from the end in $\left(\frac{1}{2}x^3+\frac{2}{x^2}\right)^9$;

(iv) $(n+1)$th term from the end in $\left(x-\frac{1}{x}\right)^{3n}$.

3. Find the middle terms in the expansion of:

(i) $(x^2+a^2)^5$; (ii) $\left(x+\frac{1}{2x}\right)^8$; (iii) $\left(3x-\frac{x^3}{6}\right)^9$; (iv) $\left(x-\frac{1}{x}\right)^{10}$;

(v) $\left(2x^2-\frac{1}{x}\right)^{10}$; (vi) $\left(x+\frac{1}{x}\right)^{2n+1}$; (vii) $(1-2x+x^2)^n$; (viii) $(1+3x+3x^2+x^3)^{2n}$;

(ix) $\left(\frac{2x^2}{3}+\frac{3}{2x^2}\right)^{10}$; (x) $\left(x+\frac{1}{x^2}\right)^{17}$.

4. Find the term independent of x in the expansion of:

(i) $\left(x^2+\frac{1}{x}\right)^9$; (ii) $\left(x^2-\frac{2}{x^3}\right)^5$; (iii) $\left(2x^4-\frac{1}{x}\right)^{10}$; (iv) $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$;

(v) $\left(x-\frac{1}{x^2}\right)^{3n}$; (vi) $\left(2x^2-\frac{1}{x}\right)^{12}$; (vii) $\left(\sqrt{\frac{x}{3}}+\frac{3}{2x^2}\right)^{10}$;

(viii) $\left(\frac{1}{2}x^{1/3}+x^{-1/5}\right)^8$; (ix) $(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$;

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6. (i) 120; (ii) –101376; (iii) No term; (iv) 680; no term;
(v) $2^8 . 3^5 . 5 . 11$ and no term; (vi) $210\ c^{12}$; (vii) ${}^{45}C_{30}$;
(viii) ${}^{2n}C_{\frac{4n-p}{3}}$; (ix) 154; (x) 60.

7. $\dfrac{228096}{x^3}$. **8.** 3. **9.** $210y^2$. **10.** $35 . 9^{13} . x^{13}$.

11. $\dfrac{9}{7}$. **12.** 1 : 32. **13.** $ab = 1$. **15.** 171.

16. 12. **17.** 7, 14. **18.** 1, 2 and 7. **19.** $-\dfrac{7}{8}$.

21. 7. **22.** 5. **23.** 9/7. **24.** 4.
25. (i) 1; (ii) 4. **26.** $-{}^{100}C_{53}$. **27.** 9. **29.** 10 or $10^{-5/2}$.
31. $r = 5$.
34. (i) 5th term, 70; (ii) 5th term, $8^7 . 5^4 . {}^{11}C_4$; (iii) 12th term; $3^5 . 5^{10} . {}^{15}C_{10}$.
37. 232. **38.** ${}^{45}C_{30}$. **41.** 1, 2.

HINTS AND SOLUTIONS

4. (ix) The term independent of x in the expansion of $(1 + x + 2x^3)\left(\dfrac{3}{2}x^2 - \dfrac{1}{3x}\right)^9$

$= $ (terms containing x^0 in the given expansion)

$+ x$ (terms containing $\dfrac{1}{x}$ in the given expansion)

$+ 2x^3$ (terms containing $\dfrac{1}{x^3}$ in the given expansion)

In the expansion of $\left(\dfrac{3}{2}x^2 - \dfrac{1}{3x}\right)^9$, $(r + 1)$th term is given by

$$t_{r+1} = {}^9C_r\left(\frac{3}{2}x^2\right)^{9-r}\left(-\frac{1}{3x}\right)^r = {}^9C_r\left(\frac{3}{2}\right)^{9-r}\cdot\left(-\frac{1}{3}\right)^r \cdot x^{18-3r}$$

Putting $r = 6$, we get the term containing x^0

$$= {}^9C_6\left(\frac{3}{2}\right)^3\cdot\left(-\frac{1}{3}\right)^6 = \left(\frac{9\cdot8\cdot7}{1\cdot2\cdot3}\cdot\frac{1}{8\cdot27}\right) = \frac{7}{18}$$

Putting $r = 7$, we get the term containing $\dfrac{1}{x^3}$

$$= {}^9C_7\left(\frac{3}{2}\right)^2\left(-\frac{1}{3}\right)^7\frac{1}{x^3} = -\left[\frac{9\cdot8}{1\cdot2}\cdot\frac{1}{4\cdot243}\right]\frac{1}{x^3} = -\frac{1}{27}\cdot\frac{1}{x^3}$$

There is no term containing $\dfrac{1}{x}$ in the given expansion.

$\therefore$ The term independent of x in the given expansion

$$= \frac{7}{18} + 2x^3\cdot\left(-\frac{1}{27}\cdot\frac{1}{x^3}\right) = \frac{7}{18} - \frac{2}{27} = \frac{17}{54}$$

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Since ${}^{n+1}C_r = {}^nC_{r-1} + {}^nC_r$

Hence the result.

31. $T_{p+1} = {}^{10}C_p(2y)^p = {}^{10}C_p 2^p y^p$

If it contains y^{r-1}, then $p = r - 1$.

$\therefore$ Coefficient of $y^{r-1} = {}^{10}C_{r-1}2^{r-1}$

i.e., $k_r = {}^{10}C_{r-1}2^{r-1}$

Changing r to $r + 2$, $k_{r+2} = {}^{10}C_{r+1}2^{r+1}$

$$\therefore \quad \frac{k_{r+2}}{k_r} = 4 \qquad \text{(Given)}$$

$$\Rightarrow \quad \frac{{}^{10}C_{r+1}\cdot 2^{r+1}}{{}^{10}C_{r-1}\cdot 2^{r-1}} = 4 \Rightarrow {}^{10}C_{r+1}\cdot 2^2 = 4.\ {}^{10}C_{r-1} \text{ or } {}^{10}C_{r+1} = {}^{10}C_{r-1}$$

$$\Rightarrow \quad \frac{(10)!}{(r+1)!(10-r-1)!} = \frac{(10)!}{(r+1)!(10-r+1)!} \Rightarrow \frac{1}{(r+1)\cdot r} = \frac{1}{(11-r)(10-r)}$$

$$\Rightarrow \quad (11-r)(10-r) = r(r+1) \Rightarrow 110 - 21r + r^2 = r^2 + r, \Rightarrow 110 = 22r, \therefore r = 5$$

33. Given ${}^nC_{r-1}$, ${}^nC_r$ and ${}^nC_{r+1}$ are in AP.

$$\therefore \quad 2{}^nC_r = {}^nC_{r-1} + {}^nC_{r+1}$$

$$\Rightarrow \quad 2 = \frac{{}^nC_{r-1}}{{}^nC_r} + \frac{{}^nC_{r+1}}{{}^nC_r} \Rightarrow 2 = \frac{r}{n-r+1} + \frac{n-r}{r+1} \qquad \left[\because \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}\right]$$

$$\Rightarrow \quad 2 = \frac{r(r+1)+(n-r)(n-r+1)}{(r+1)(n-r+1)}$$

$$\Rightarrow \quad 2\{(n-r+1)(r+1)\} = r(r+1) + (n-r)(n-r+1)$$

$$\Rightarrow \quad 2nr - 2r^2 + 2n + 2 = r^2 + r + n^2 - 2nr + r^2 + n - r$$

$$\Rightarrow \quad n^2 - 4nr - n + 4r^2 - 2 = 0 \Rightarrow n^2 - n(4r+1) + 4r^2 - 2 = 0$$

36. The given expression $= \left\{\left(x+\frac{1}{x}\right)^2\right\}^n = \left(x+\frac{1}{x}\right)^{2n}$

The number of terms $= 2n + 1$, which is odd.

$$\text{The middle term} = t_{\frac{(2n+1)+1}{2}} = t_{n+1} = {}^{2n}C_n x^{2n-n}\cdot\left(\frac{1}{x}\right)^n = \frac{(2n)!}{(n)!(n)!}$$

$$= \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6...(2n-1)\cdot 2n}{(n)!(n)!} = \frac{[1\cdot 3\cdot 5...(2n-1)][2\cdot 4\cdot 6...2n]}{(n)!(n)!}$$

$$= \frac{[1\cdot 3\cdot 5...(2n-1)]\,2^n[1\cdot 2\cdot 3...n]}{(n)!\,1\cdot 2\cdot 3....n} = \frac{1\cdot 3\cdot 5...(2n-1)}{(n)!}\cdot 2^n$$

37. Coefficient of x^{-1} in $(1 + 3x^2 + x^4)\left(1+\frac{1}{x}\right)^8$

$= \text{Coefficient of } x^{-1} \text{ in } \left(1+\frac{1}{x}\right)^8 + \text{Coefficient of } x^{-1} \text{ in}$

$3x^2\left(1+\frac{1}{x}\right)^8 + \text{Coefficient of } x^{-1} \text{ in } x^4\left(1+\frac{1}{x}\right)^8$

PROGRESSIONS

If the terms of a sequence follow certain pattern, then the sequence is called a *progression.*

Illustration 1. Consider the following sequences:

(i) 3, 5, 7, 9, ..., 21; (ii) 8, 5, 2, – 1, – 4, ...;

(iii) 2, 6, 18, 54, ..., 1458; (iv) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$; (v) 1, 4, 9, 16,... .

We observe that each term (except first) in (i) is formed by adding 2 to the preceding term; each term in (ii) is formed by subtracting 3 from the preceding term; each term in (iii) is formed by multiplying the preceding term by 3; each term in (iv) is formed by dividing the preceding term by 2; each term in (v) is formed by squaring the next natural number. Thus, each of (i) to (v) is a progression. Moreover, (i) and (iii) are finite sequences whereas (ii), (iv) and (v) are infinite sequences.

However, to define a sequence we need not always have an explicit formula for the *nth* term. For example, for the infinite sequence 2, 3, 5, 7, 11, 13, 17, ... of all positive prime numbers, we may not be able to give an explicit formula for the *nth* term.

SERIES

By adding or subtracting the terms of a sequence, we obtain a *series.* A series is finite or infinite according as the number of terms in the corresponding sequence is finite or infinite.

Illustration 2. The following:

(i) 3 + 5 + 7 + 9 + ... + 21; (ii) 8 + 5 + 2 + (– 1) + ...;

(iii) 2 + 6 + 18 + 54 + ... + 1458; (iv) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$;

(v) 1 + 4 + 9 + 16 + ...

are the series corresponding to the earlier sequences, (i) to (v).

Example 1. *Write the first four terms of the sequences defined by*

(i) $t_n = 4n^2 + 3$; (ii) $t_n = \frac{3n}{n+4}$.

Solution: (i) We have,

$$t_n = 4n^2 + 3$$

Putting $n = 1, 2, 3, 4$, we get

$$t_1 = 4 \cdot 1^2 + 3 = 7,\ t_2 = 4 \cdot 2^2 + 3 = 19,\ t_3 = 4 \cdot 3^2 + 3 = 39,$$
$$t_4 = 4 \cdot 4^2 + 3 = 67.$$

Thus, the first four terms of the sequence are 7, 19, 39, and 67.

(ii) We have,

$$t_n = \frac{3n}{n+4}$$

Putting $n = 1, 2, 3, 4$, we get

$$t_1 = \frac{3.1}{1+4} = \frac{3}{5},\ t_2 = \frac{3.2}{2+4} = 1,\ t_3 = \frac{3.3}{3+4} = \frac{9}{7},\ t_4 = \frac{3.4}{4+4} = \frac{3}{2}.$$

Thus, the first four terms of the given sequence are $\frac{3}{5}$, 1, $\frac{9}{7}$ and $\frac{3}{2}$.

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Solution: Clearly, the given sequence is an AP with $a = 5$ and $d = -3$.

$\therefore \quad t_n = a + (n-1)\,d = 5 + (n-1)(-3) = -3n + 8.$

For the 19th term, putting $n = 19$, we get $t_{19} = -3.19 + 8 = -49$.

Example 3. *Which term of the sequence –3, –7, –11, –15,... is –403? Also find which term, if any, of the given sequence is –500?*

Solution: Clearly, the given sequence is an AP with $a = -3$ and $d = -4$.

Let -403 be the nth term of the sequence, then

$-403 = (-3) + (n-1)(-4)$ $\quad [\because t_n = a + (n-1)d]$

$\Rightarrow \quad 4(n-1) = 400 \Rightarrow n - 1 = 100 \Rightarrow n = 101$

$\therefore$ It is the 101st term.

Let –500 be the mth term of the given sequence, then

$$-500 = (-3) + (m-1)(-4) \Rightarrow 4(m-1) = 500 - 3 \Rightarrow 4m = 497 + 4 \Rightarrow m = \frac{501}{4},$$

which is not a natural number. Therefore, –500 is not a term of the given sequence.

Example 4. *If the 9th term of an AP is zero, prove that 29th term is double its 19th term.*

Solution: Let a be the first term and d, the common difference of the given AP

Given: $\quad t_9 = 0 \;\Rightarrow\; a + (9-1)d = 0 \;\Rightarrow\; a + 8d = 0 \;\Rightarrow\; a = -8d \quad$ ...(1)

Now, $\quad \dfrac{t_{29}}{t_{19}} = \dfrac{a + (29-1)d}{a + (19-1)d} = \dfrac{a + 28d}{a + 18d} = \dfrac{-8d + 28d}{-8d + 18d} \quad$ [Using (1)]

$\Rightarrow \quad \dfrac{t_{29}}{t_{19}} = \dfrac{20d}{10d} = 2 \Rightarrow t_{29} = 2t_{19}.$

Example 5. *If each term in an AP is doubled. Is the resultant sequence also an AP? If so, write its first term, common difference and the general term.*

Solution: Let the given AP be a, $a + d$, $a + 2d$, ... The sequence obtained by multiplying each term by 2 is $2a$, $2(a + d)$, $2(a + 2d)$,... which is clearly an AP with first term $2a$ and common difference $2d$.

General term $= 2a + (n-1)2d = 2[a + (n-1)d]$

Example 6. *If the third term of an AP is* 18 *and the seventh term is* 30*, find the series.*

Solution: Let a be the first term and d, the common difference of the given *AP*.

Given: $\quad t_3 = 18$, i.e., $a + 2d = 18 \quad$ ...(1)

and $\quad t_7 = 30$, i.e., $a + 6d = 30 \quad$ ...(2)

Subtracting (1) from (2), we get $4d = 12 \Rightarrow d = 3$. Substituting this value of d in (1), we get

$$a + 2.3 = 18 \;\Rightarrow\; a = 12$$

$$t_n = 12 + (n-1)\,3 = 3n + 9$$

Putting $n = 1, 2, 3, 4, \ldots$, the AP is 12, 15, 18, 21,... Therefore, the required series is $12 + 15 + 18 + 21 + \cdots$

Example 7. *The 2nd, 31st and last term of AP are* $7\frac{3}{4}$, $\frac{1}{2}$ *and* $-6\frac{1}{2}$, *respectively. Find the first term and the number of terms.*

(i) t_n is purely real if imaginary part is 0.

$\therefore$ From (1),

$$-(8-2n)=0 \text{ or } -8+2n=0 \text{ or } 2n=8 \text{ or } n=4$$

$\therefore$ t_4 is purely real.

(ii) t_n is purely imaginary if real part is 0.

$\therefore$ From (1),

$$9-n=0 \text{ or } n=9$$

$\therefore$ t_9 is purely imaginary.

Example 20. *If the pth, qth and rth terms of an AP are a, b, c, respectively, prove that* $a(q-r)+b(r-p)+c(p-q)=0$.

Solution: Let A be the first term and D, the common difference of the AP

Given: $t_p = a \Rightarrow A+(p-1)D = a$ ...(1)

$t_q = b \Rightarrow A+(q-1)D = b$ ...(2)

$t_r = c \Rightarrow A+(r-1)D = c$ ...(3)

$\therefore$ $a(q-r)+b(r-p)+c(p-q)$

$=[A+(p-1)D](q-r)+[A+(q-1)D](r-p)+[A+(r-1)D](p-q)$

[Using (1), (2) and (3)]

$=(q-r+r-p+p-q)A+[(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)]D$

$=0\cdot A+(pq-pr-q+r+qr-pq-r+p+pr-p-qr+q)D$

$=0\cdot A+0\cdot D=0$

Example 21. *The sum of three consecutive terms of an AP is 15 and their product 105. Find the numbers.*

Solution: Let the three numbers in AP be $a-d,\ a,\ a+d$.

Given: Sum of numbers = 15

i.e., $(a-d)+a+(a+d)=15$ or $3a=15$, $\therefore$ $a=5$

Also Product = 105

i.e., $(a-d)\,a\,(a+d)=105$

or $(5-d)\cdot 5\cdot(5+d)=105$ [Putting the value of a]

or $25-d^2=21$ or $d^2=4$, $\therefore d=\pm 2$

When $d=2$, the three numbers in AP are 3, 5, 7.

When $d=-2$, the three numbers in AP are 7, 5, 3.

Example 22. *The sum of four numbers in AP is 20 and the sum of their squares is 120. Find the numbers.*

Solution: Let the four numbers in AP be $a-3d,\ a-d,\ a+d,\ a+3d$.

Given: Sum of numbers = 20

i.e., $a-3d+a-d+a+d+a+3d=20$ or $4a=20$, $\therefore$ $a=5$

Also, Sum of their squares = 120

i.e., $(a-3d)^2+(a-d)^2+(a+d)^2+(a+3d)^2=120$

or $a^2+9d^2+a^2+d^2=60$ or $2a^2+10d^2=60$

or $50+10d^2=60$ [Putting the value of a]

18. Let a_1 be the first term and d, the common difference of the AP

$\therefore \quad t_{m+n} = a_1 + (m+n-1)d$ ...(1)

$t_{m-n} = a_1 + (m-n-1)d$ ...(2)

$t_m = a_1 + (m-1)d$ ...(3)

Add (1) and (2).

19. Let three parts of 69 be $(a-d)$, a and $(a+d)$.

$\therefore \quad (a-d) + a + (a+d) = 69 \Rightarrow 3a = 69 \Rightarrow a = 23$

Thus, the parts are $(23-d)$, 23 and $(23+d)$.

As per question: Product of two smaller parts = 483

$\therefore \quad (23-d)(23) = 483 \Rightarrow 23 - d = \dfrac{483}{23} = 21 \Rightarrow d = 23 - 21 = 2$

$\therefore$ The parts are $(23-2)$, 23 and $(23+2)$, i.e., 21, 23 and 25

Hence, 69 is splited into 3 parts viz, 21, 23 and 25.

21. Let a_1, be the first term and d, the common difference. Here $n = 21$. The three middle terms are a_{10}, a_{11}, a_{12}.

Given $\quad a_{10} + a_{11} + a_{12} = 129 \Rightarrow (a_1 + 9d) + (a_1 + 10d) + (a_1 + 11d) = 129$

$\Rightarrow 3a_1 + 30d = 129$ or $a_1 + 10d = 43$ ...(1)

The last three terms are a_{19}, a_{20}, a_{21}.

Given $\quad a_{19} + a_{20} + a_{21} = 237 \Rightarrow (a_1 + 18d) + (a_1 + 19d) + (a_1 + 20d) = 237$

$\Rightarrow 3a_1 + 57d = 237$ or $a_1 + 19d = 79$ ...(2)

Solving (1) and (2), we get a_1 and d and hence the series.

22. In Example 20, subtracting (2) from (1), (3) from (2) and (1) from (3), we get

$$a - b = (p-q)D,\ b - c = (q-r)D,\ c - a = (r-p)D$$

$\therefore \quad (a-b)r + (b-c)p + (c-a)q = (p-q)Dr + (q-r)Dp + (r-p)Dq$

$= D[(p-q)r + (q-r)p + (r-p)q] = 0$

23. Given: $\quad a + (m-1)d = \dfrac{1}{n}$ ...(1) and $a + (n-1)d = \dfrac{1}{m}$ ...(2)

Subtracting (2) from (1), we get

$$\frac{1}{n} - \frac{1}{m} = (m-n)d \Rightarrow d = \frac{1}{mn}$$

Putting $d = \dfrac{1}{mn}$ in (1), we get

$$a + (m-1).\frac{1}{mn} = \frac{1}{n} \Rightarrow a = \frac{1}{mn}$$

26. The numbers between 100 and 500 that are divisible by 7 are 105, 112, 119, 126, 133, ..., 483, 490, 497. Let such numbers be n. Then

$$497 = 105 + (n-1)7 \quad \text{or} \quad n = 57$$

The numbers between 100 and 500 that are divisible by 21 are 105, 126 147, ..., 483. Let such numbers be m. Then, $483 = 105 + (m-1)21$ or $m = 19$.

Hence, the required number $= n - m = 57 - 19 = 38$.

27. If possible let the mth term of the first AP be identical with nth term of the second AP

$\therefore \quad 2 + (m-1)3 = 3 + (n-1)2$ or $3m - 1 = 2n + 1$

or $\quad 3m - 6 = 2n - 4$ or $3(m-2) = 2(n-2)$

or $\quad \dfrac{m-2}{2} = \dfrac{n-2}{3} = k$ (say) $\quad \therefore \quad m = 2k + 2$ and $n = 3k + 2$

But $\quad 1 \le m \le 60$ and $1 \le n \le 50$

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We have

$$t_6 = 3 \Rightarrow \frac{1}{2} + (6-1)d = 3 \Rightarrow 3 - \frac{1}{2} = 5d \text{ or } 5d = \frac{5}{2} \text{ or } d = \frac{1}{2}$$

$\therefore$
$$A_1 = a + d = \frac{1}{2} + \frac{1}{2} = 1$$

$$A_2 = a + 2d = \frac{1}{2} + 2 \times \frac{1}{2} = \frac{1}{2} + 1 = 1\frac{1}{2}$$

$$A_3 = a + 3d = \frac{1}{2} + 3 \times \frac{1}{2} = \frac{1}{2} + \frac{3}{2} = \frac{4}{2} = 2$$

$$A_4 = a + 4d = \frac{1}{2} + 4 \times \frac{1}{2} = \frac{1}{2} + 2 = 2\frac{1}{2}$$

Also,
$$A_1 + A_2 + A_3 + A_4 = 1 + 1\frac{1}{2} + 2 + 2\frac{1}{2} = 7$$

Example 8. *Prove that the sum of m AMs between any two numbers: sum of n AMs between them is m : n.*

Solution: Let the two numbers be a and b. Then sum of m AMs between a and b

$$= m \text{ (AM between } a \text{ and } b) = m\left(\frac{a+b}{2}\right) \quad \text{...(1)}$$

Also, sum of n AMs between a and b

$$= n \text{ (AM between } a \text{ and } b) = n\left(\frac{a+b}{2}\right) \quad \text{...(2)}$$

$$\therefore \quad \frac{\text{Sum of } m \text{ AMs}}{\text{Sum of } n \text{ AMs}} = \frac{m\left(\frac{a+b}{2}\right)}{n\left(\frac{a+b}{2}\right)} = \frac{m}{n}$$

EXERCISE 9.5

LEVEL OF DIFFICULTY A

1. Find the AM of
   (i) –5 and 41; (ii) $3p - 2q$ and $3p + 2q$; (iii) $(a+b)^2$ and $(a-b)^2$.
2. Insert three AMs between 5 and 29.
3. Insert five AMs between $4\frac{1}{2}$ and $1\frac{1}{2}$.
4. The n AMs between 20 and 80 are such that the first mean: the last mean = 1 : 3, find the value of n.
5. If n AMs be inserted between 1 and 51 such that 4th mean: 7th mean = 3 : 5, find n.
6. If 11 AMs are inserted between 28 and 10, find the middle term of the AP.
7. Between 1 and 31, m-arithmetic means have been inserted in such a way that the ratio of the 7th and $(m - 1)$th means is 5 : 9. Find the value of m.
8. Show that the sum of n-arithmetic means between two numbers is n times the single arithmetic mean between them.

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24. Show that the product of n terms, where n is odd, of a GP will be equal to nth power of the middle term.

25. Three numbers whose sum of 70 are in GP. If each of the extremes is multiplied by 4 and the mean by 5, the numbers will be in AP Find the numbers.

26. The sum of first three terms of a GP is 7 and the sum of their squares is 21. Determine the first five terms of the GP.

27. If pth and qth terms of a GP are q and p respectively, show that the $(p+q)$th term is $(q^p/p^q)^{1/(p-q)}$.

28. If a b, c are in AP as well as in GP, show that $a = b = c$.

29. If x, y, z are in GP and $a^x = b^y = c^z$, show that $\log_b a = log_c b$.

30. If $a^{1/x} = b^{1/y} = c^{1/z}$ and a, b, c are in GP, show that x, y, z are in IE,

31. The three numbers a, b, c between 2 and 18 are such that their sum is 25; the numbers 2, a, b are consecutive terms of an AP and the numbers b, c, 18 are consecutive terms of a GP. Find the three numbers.

32. If a, b, c, d are non-zero real numbers such that $(a^2+b^2+c^2)(b^2+c^2+d^2) \le (ab+bc+cd)^2$, show that a, b, c, d are in GP.

33. If a, b, c, d and p are distinct real numbers such that $(a^2+b^2+c^2)p^2 - 2(ab+bc+cd)p + (b^2+c^2+d^2) \le 0$,
show that a, b, c, d are in GP.

34. If $a_1, a_2, a_3, \ldots$ are in AP and a_p, a_q, a_r are in GP, show that $a_q : a_p = (r-p) : (q-p)$

35. If a, b, c are the pth, qth and rth terms respectively of an AP and GP both, show that the product of the roots of equation $a^b b^c c^a x^2 - a\,b\,c\,x + a^c b^a c^b = 0$ is equal to 1.

36. If a, b, c, d, e, x are real and $(a^2+b^2+c^2+d^2)x^2 - 2\ (ab+bc+cd+de)x + (b^2+c^2+d^2+e^2) \le 0$,
show that a, b, c, d, e are in GP.

37. If p, q, r are in one GP and a, b, c in another GP, show that cp, bq, ar are in GP.

38. If a, b, c are in GP and $\log_c a$, $\log_b c$ and $\log_a b$ are in AP, show that the common difference of AP is 3/2.

39. If $\log\left(\frac{2b}{3c}\right)$, $\log\left(\frac{4c}{9a}\right)$ and $\log\left(\frac{8a}{27b}\right)$ are in AP, where a, b, c are in GP; show that a, b, c are the lengths of sides of an equilateral triangle.

40. If a, b, c are in AP, show that $2^{ax+1}, 2^{bx+1}, 2^{cx+1}$ are in GP.

41. If a, b, c, d are in GP, show that $(a^3+b^3)^{-1}, (b^3+c^3)^{-1}, (c^3+d^3)^{-1}$ are in GP.

Answers

1. (i) -512; (ii) $\frac{5}{2^{20}}, \frac{5}{2^n}$; (iii) $3^{\frac{-27}{2}}$; (iv) $(-1)^{n-1} \cdot 2^{\frac{3n-4}{2}}$ -2^{16}.

2. 3072. 3. $\frac{4}{3}$. 4. 3, 2.

5. (i) 6th; (ii) 12th; (iii) 9th; (iv) 12th; (v) 9th; (vi) 13th.

7. 10. 8. ± 1. 9. -2187. 10. 6144.

12. $-\frac{1}{2}$. 15. 5, 10, 20, ... or 20, 10, 5, ...

16. 8, 16, 32 or 32, 16, 8. 17. 3. 18. 2, 5, 8 or 26, 5, $-$ 16.

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Solution: Let r be the common ratio of the GP, then we have $b = ar$, $c = ar^2$ and $d = ar^3$.

(i) We have

$$a + b = a + ar = a(1 + r)$$
$$b + c = ar + ar^2 = ar(1 + r)$$
$$c + d = ar^2 + ar^3 = ar^2(1 + r)$$

Therefore, $\frac{b+c}{a+b} = \frac{ar(1+r)}{a(1+r)} = r$ or $b + c = (a + b)r$

and $\frac{c+d}{b+c} = \frac{ar^2(1+r)}{ar(1+r)} = r$ or $c + d = (b + c)r = (a + b)r^2$

Hence, $a + b$, $b + c$ and $c + d$ are also in GP, with the same common ratio.

(ii) We have

$$a^2 - b^2 = a^2 - a^2r^2 = a^2(1 - r^2)$$
$$b^2 - c^2 = a^2r^2 - a^2r^4 = a^2r^2(1 - r^2)$$
$$c^2 - d^2 = a^2r^4 - a^2r^6 = a^2r^4(1 - r^2)$$

Therefore, $\frac{b^2 - c^2}{a^2 - b^2} = \frac{a^2 - r^2(1 - r^2)}{a^2(1 - r^2)} = r^2$ or $b^2 - c^2 = (a^2 - b^2)r^2$

and $\frac{c^2 - d^2}{b^2 - c^2} = \frac{a^2r^4(1 - r^2)}{a^2r^2(1 - r^2)} = r^2$ or $c^2 - d^2 = (b^2 - c^2)r^2 = (a^2 - b^2)r^4$

Hence, $a^2 - b^2$, $b^2 - c^2$ and $c^2 - d^2$ are in GP with the common ratio r^2.

(iii) We have

$$a^2 + b^2 + c^2 = a^2 + a^2r^2 + a^2r^4 = a^2(1 + r^2 + r^4)$$
$$ab + bc + cd = a^2r + a^2r^3 + a^2r^5 = a^2r(1 + r^2 + r^4)$$
$$b^2 + c^2 + d^2 = a^2r^2 + a^2r^4 + a^2r^6 = a^2r^2(1 + r^2 + r^4)$$

Therefore, $\frac{ab + bc + cd}{a^2 + b^2 + c^2} = \frac{a^2r(1 + r^2 + r^4)}{a^2(1 + r^2 + r^4)} = r$

or $ab + bc + cd = r(a^2 + b^2 + c^2)$

and $\frac{b^2 + c^2 + d^2}{ab + bc + cd} = \frac{a^2r^2(1 + r^2 + r^4)}{a^2r(1 + r^2 + r^4)} = r$

or $b^2 + c^2 + d^2 = (ab + bc + cd)r = (a^2 + b^2 + c^2)r^2$

Hence, $a^2 + b^2 + c^2$, $ab + bc + cd$ and $b^2 + c^2 + d^2$ are in GP with the same common ratio r.

(iv) We have

$$\frac{1}{a^2 + b^2} = \frac{1}{a^2 + a^2r^2} = \frac{1}{a^2(1 + r^2)}$$
$$\frac{1}{b^2 + c^2} = \frac{1}{a^2r^2 + a^2r^4} = \frac{1}{a^2r^2(1 + r^2)}$$
$$\frac{1}{c^2 + d^2} = \frac{1}{a^2r^4 + a^2r^6} = \frac{1}{a^2r^4(1 + r^2)}$$

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$$x = \frac{1.(3) + 3.1}{1+3} = 0 \quad \text{and} \quad y = \frac{1.9 + 3.(-3)}{1+3} = 0$$

III. In particular, if $m = n$, the coordinates of the mid-point of the line segment joining the points (x_1, y_1) and (x_2, y_2) are

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

IV. Area of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\Delta = \frac{1}{2} | x_1(y_2 - y_3) + x_2(y_3 - y_1) + x(y_1 - y_2) |$$

For example, the area of the triangle, whose vertices are (4, 4), (3, –2) and (–3, 16) is

$$\Delta = \frac{1}{2} | 4(-2-16) + 3(16-4) + (-3)(4+2) | = \frac{|-54|}{2} = 27$$

V. If the area of the triangle ABC is zero, then three points A, B and C lie on a line, i.e., they are collinear.

SLOPE OR GRADIENT OF A LINE

The *tangent of the angle* which a line makes with the *positive direction of x-axis is* called the *slope or the gradient* of the line. It is generally denoted by m. [Refer Figs. 10.1(a) and (b).

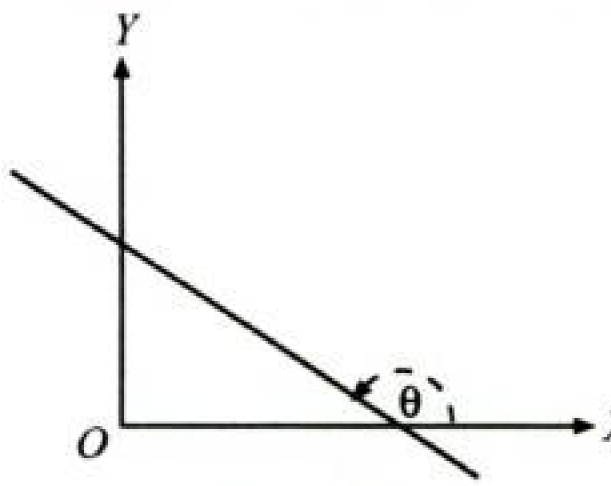

Fig. 10.1(a)

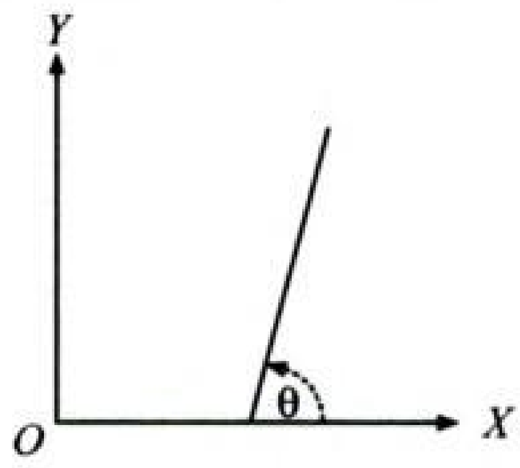

Fig. 10.1(b)

Thus, if a line makes an angle θ with x-axis, then its slope = tan θ, i.e. $m = \tan\theta$. If a line is parallel to x-axis, $\theta = 0^\circ$

$\therefore \quad m = \tan 0^\circ = 0$

If a line is parallel to y-axis, it is perpendicular to x-axis so that $\theta = 90^\circ$. Therefore, $m = \tan 90^\circ = \infty$. That is why we do not define the slope of a vertical line.

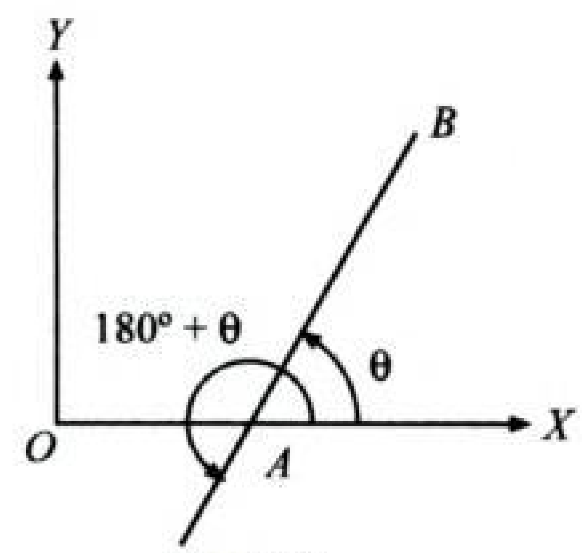

Fig. 10.2

CAUTION

Slope of a line segment is independent of its sense, for Slope of $BA = \tan(180^\circ + \theta) = \tan\theta$ = Slope of AB. [see Fig. 10.2]

Slope of the Line Joining Two Given Points

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two given points. Let θ be the inclination of the line PQ with x-axis so that $m = \tan\theta$.

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6. If the straight line $y = mx + c$ passes through the points (2, 4) and (–3, 6), find the values of m and c.

7. Find the equation of the straight line which makes an angle α with the positive direction of the x-axis, where $\alpha = \sin^{-1}\left(\frac{3}{5}\right)$ and makes an intercept –4 on the y-axis.

8. Find the equations of the lines through the point (0, 2) and making an angle $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ with x-axis. Also, find the lines parallel to them cutting the y-axis at a distance 2 below the origin. Find also their point of intersection with x-axis.

9. Find the equation of the line having y-intercept –5 and parallel to the join of (3, 7) and (–2, 0).

10. Find the equation of the line having y-intercept –5 and perpendicular to the straight line joining (–1, 6) and (–2, –3).

11. Find the equation of the line which cuts off an intercept 8 from the x-axis and makes an angle of 60° with the positive direction of y-axis.

LEVEL OF DIFFICULTY B

12. Find the equation of a straight line which is equidistant from the lines $x = -7$ and $x = 3$.

13. Find the equation of the bisectors of the angle between the coordinate axes.

14. If the point A is symmetric to the point $B(4, -1)$ w.r.t. the bisector of the first quadrant, then find the length of AB.

15. If $P(1, 0)$, $Q(-1, 0)$, $R(2, 0)$ are three given points, then find the locus of the point S satisfying the relation $SQ^2 + SR^2 = 2SP^2$.

16. The co-ordinates of the vertices A and B, i.e., isosceles triangle ABC ($AC = BC$) are (– 2, 3) and (2, 0) respectively. If a line, parallel to AB and having a y-intercept equal to $\frac{43}{12}$, passes through C, then find the coordinates of C.

17. The points (1, 3) and (5, 1) are the opposite vertices of a rectangle. The other two vertices lie on the line $y = 2x + c$. Find c and the remaining vertices.

Answers

1. $y = x$.

2. (i) $3x + 4y = 0$; (ii) $2x - y + 1 = 0$; (iii) $5x - y + 4 = 0$; (iv) $2x + y - 3 = 0$.

3. $x - \sqrt{3}y - 2\sqrt{3} = 0$. 4. $y = \frac{3}{4}x - 2$.

5. $y = \sqrt{3}x + 3$. 6. $m = -\frac{2}{5}$; $c = \frac{24}{5}$.

7. $3x - 4y - 16 = 0$ and $3x + 4y + 16 = 0$.

8. $y = \sqrt{3}x + 2$, $y = -\sqrt{3}x + 2$, $y = \sqrt{3}x - 2$,
$y = -\sqrt{3}x - 2$; $\left(\frac{2}{\sqrt{3}}, 0\right), \left(-\frac{2}{\sqrt{3}}, 0\right)$.

9. $7x - 5y - 25 = 0$.

10. $x + 9y + 45 = 0$. 11. $x - \sqrt{3}y = 8$. 12. $x = -2$.

13. $x - y = 0$; $x + y = 0$. 14. $5\sqrt{2}$. 15. $x = -\frac{3}{2}$.

16. $\left(1, \frac{17}{6}\right)$. 17. –4; (4, 4) and (2, 0).

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23. Find the direction in which a straight line must be drawn through the point (–1, 2) so that its point of intersection with the line $x + y = 4$ may be at a distance of 3 units from this point.

24. Prove that the lines $2x + 3y = 19$ and $2x + 3y + 7 = 0$ are equidistant from the line $2x + 3y = 6$.

25. Find the points on the x-axis, whose distances from the line $\frac{x}{3} + \frac{y}{4} = 1$ is 4 units.

26. Find the value of λ so that the distance of the point (4, 1) from the line $3x - 4y + \lambda = 0$ is 4 units.

27. Find the length of the perpendicular from the point (4, –7) to the line joining the origin and the point of intersection of the lines $2x - 3y + 14 = 0$ and $5x + 4y - 7 = 0$.

LEVEL OF DIFFICULTY B

28. If $5x - 12y - 65 = 0$ and $5x - 12y + 26 = 0$ are the equations of a pair of opposite sides of a square, find the area of the square.

29. Find the equation of a locus of a point P which is equidistant from the straight line $3x - 4y + 2 = 0$ and the origin.

30. The perpendicular distance of a line from the origin is 5 units and its slope is –1. Find the equation of the line.

31. Find the sum of the abscissas of all the points on the line $x + y = 4$ that lie at a unit distance from the line $4x + 3y - 10 = 0$.

32. Find the equation of the straight line that can be drawn through the point (4, –5) so that its distance from the point (–2, 3) is equal to 12.

33. Find the equation of the straight line which cuts off intercept on x-axis twice that on y-axis and is at a unit distance from the origin.

34. On the portion of the straight line $x + y = 2$ which is intercepted between the axes, a square is constructed away from the origin, with this portion as one of its side. If p denotes the perpendicular distance of a side of this square from the origin, then find the maximum value of p.

35. Prove that the parallelogram formed by the lines $\frac{x}{a} + \frac{y}{b} = 1$, $\frac{x}{b} + \frac{y}{a} = 1$, $\frac{x}{a} + \frac{y}{b} = 2$ and $\frac{x}{b} + \frac{y}{a} = 2$ is a rhombus.

36. Base of an equilateral triangle lies along the line $9x + 40y - 50 = 0$ and its vertex lies on the line $9x + 40y + 32 = 0$. Find the length of each side of the triangle and also find its area.

37. Prove that the origin lies inside the triangle whose vertices are (4, 5), (–4, 3) and (–1, –3).

38. A points P is such that the sum of squares of its distances from the two axes of coordinates is equal to the square of its distance from the line $x - y = 1$. Find the equation of the locus of P.

39. Find the length of the perpendicular from the origin to the line joining two points whose coordinates are $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$.

40. Find the points on the y-axis whose perpendicular distance from the line $4x - 3y - 12 = 0$ is 3.

41. A straight line is parallel to the line $3x - y - 3 = 0$ and $3x - y + 5 = 0$ and lies between them. Find the equation of the line if its distances from there lines are in the ratio 3 : 5.

Rotating a right triangle around one of its shorter sides (making the side the axis) will produce a right circular cone. In an oblique circular cone (right) the axis does not form a 90° angle with the directrix.

CONIC SECTIONS—AS SECTIONS OF A RIGHT CIRCULAR CONE

1. Section of a right circular cone by a plane, which is passing through its vertex, is a *pair of straight lines*. Lines always pass through the vertex of the cone (See Fig. 11.2).

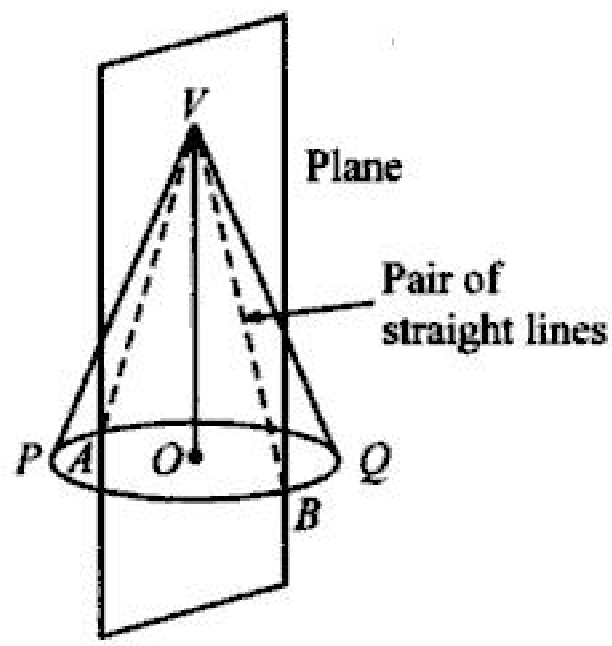

Fig. 11.2

2. Section of a right circular cone by a plane, which is parallel to its base, is a *circle* (Fig. 11.3).

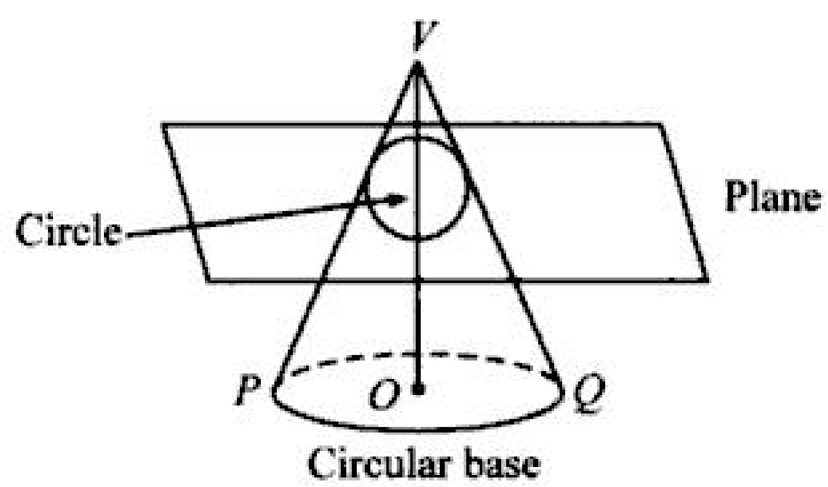

Fig. 11.3

3. Section of a right circular cone by a plane, which is parallel to a generator of the cone, is a *parabola* (Refer Fig. 11.4).

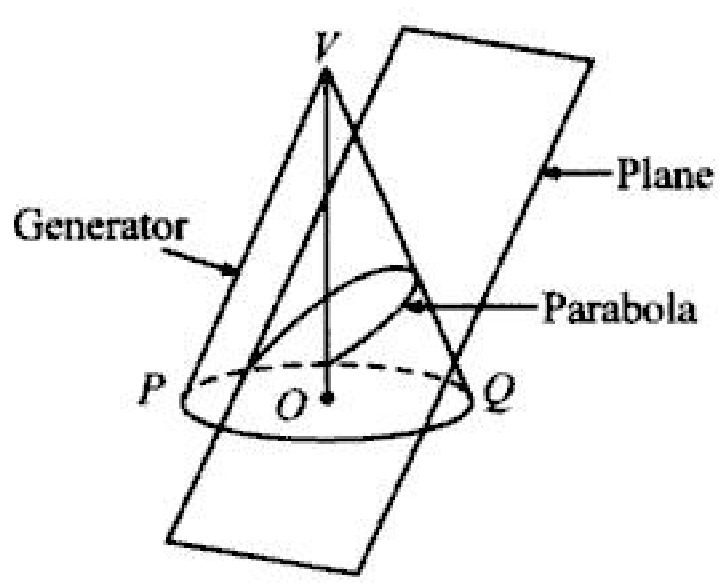

Fig. 11.4

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Let $P(x_1, y_1)$ be any point on the hyperbola. Let PN be perpendicular to the x-axis. Then

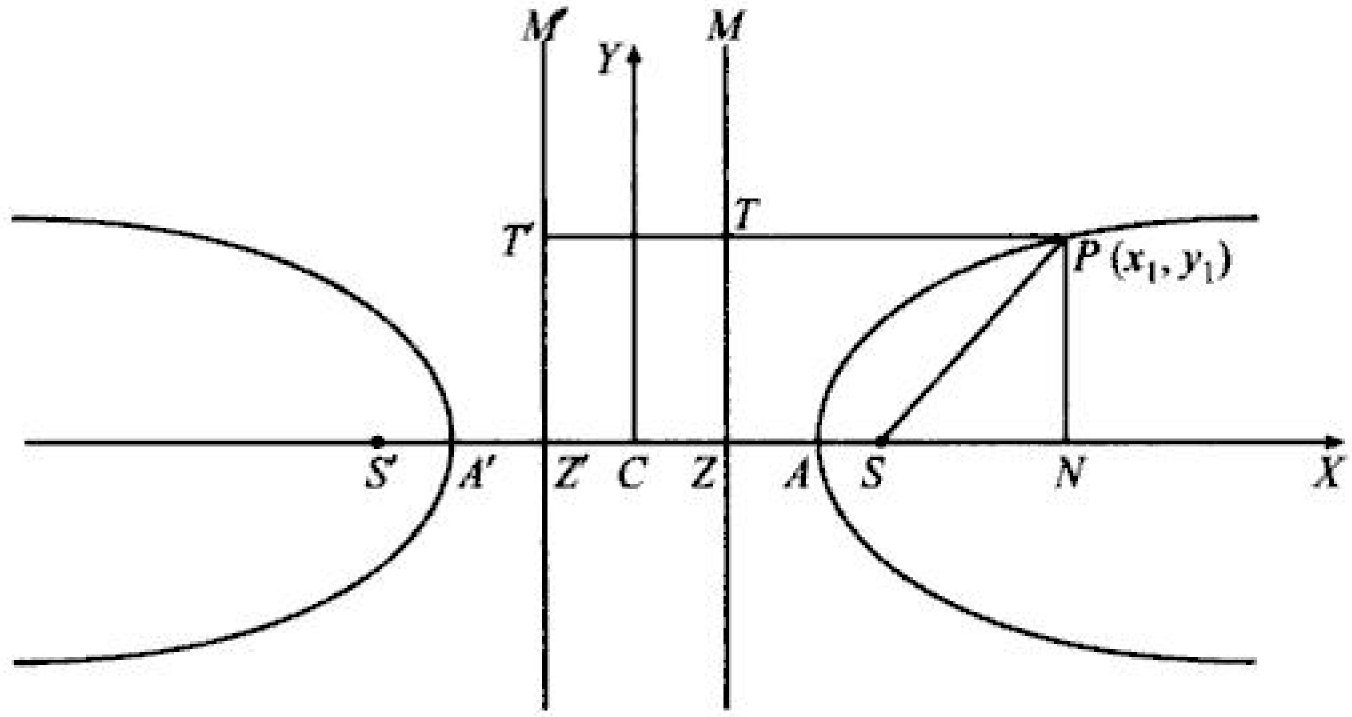

Fig. 11.75

$$SP = e \times PT = e \times ZN = e(CN - CZ) = e\left(x_1 - \frac{a}{e}\right) = ex_1 - a$$

$$S'P = e \times PT' = e \times Z'N = e(CZ' + CN) = e\left(x_1 + \frac{a}{e}\right) = ex_1 + a$$

$\therefore \quad S'P - SP = (ex_1 + a) - (ex_1 - a) = 2a = AA'$ (Transverse axis)

So, we can also define hyperbola as follows:

Second Definition of Hyperbola

A hyperbola is the locus of a point, the difference of whose distances from two fixed points is a positive constant.

Equation of Hyperbola from Second Definition

$$|PS - PS'| = 2a \Rightarrow PS - PS' = \pm 2a$$

$$\therefore \quad \sqrt{(x-ae)^2 + y^2} - \sqrt{(x+ae)^2 + y^2} = \pm 2a$$

$$\Rightarrow \quad \sqrt{(x-ae)^2 + y^2} = \pm 2a + \sqrt{(x+ae)^2 + y^2}$$

Squaring both sides, we have

$$(x - ae)^2 + y^2 = 4a^2 + (x + ae)^2 + y^2 \pm 4a\sqrt{(x+ae)^2 + y^2}$$

$$\Rightarrow \quad -4aex - 4a^2 = \pm 4a\sqrt{(x+ae)^2 + y^2}$$

Dividing both sides by $-4a$, we have

$$ex + a = \mp\sqrt{(x+ae)^2 + y^2}$$

Squaring both sides again, we have

$$e^2x^2 + 2aex + a^2 = x^2 + 2aex + a^2e^2 + y^2 \Rightarrow (e^2 - 1)x^2 - y^2 = a^2(e^2 - 1)$$

Dividing both sides by $a^2(e^2 - 1)$, we get $\dfrac{x^2}{a^2} - \dfrac{y^2}{a^2(e^2-1)} = 1.$

Since $e^2 - 1 > 0$, we put $b = a\sqrt{e^2 - 1}$. Then the above equation becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

which is the required equation of the hyperbola in standard form.

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(iii) Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$\therefore$ Transverse axis = $2a = 7$ $\therefore$ $a = \frac{7}{2}$

Equation of the hyperbola is $\frac{x^2}{\frac{49}{4}} - \frac{y^2}{b^2} = 1$. It passes through the point (5, –2),

$$\therefore \quad \frac{25}{\frac{49}{4}} - \frac{4}{b^2} = 1 \quad \therefore \quad \frac{100}{49} - \frac{4}{b^2} = 1 \Rightarrow \frac{51}{49} = \frac{4}{b^2} \quad \therefore \quad b^2 = \frac{196}{51}$$

The equation of the hyperbola is:

$$\frac{x^2}{\frac{49}{4}} - \frac{y^2}{\frac{196}{51}} = 1, \quad \text{i.e.,} \quad \frac{4x^2}{49} - \frac{51y^2}{196} = 1$$

Example 16. *Find whether the points* (1, 2) and (6, 3) lie inside or outside the hyperbola $4x^2 - 9y^2 - 8x + 36y - 68 = 0$.

Solution: Let $(x_1, y_1) = (1, 2)$. Since

$$4x_1^2 - 9y_1^2 - 8x_1 + 36y_1 - 68 = 4 - 36 - 8 + 72 - 68 = -36 < 0$$

the point (1, 2) lies inside the hyperbola.

Let $(x_1, y_1) = (6, 3)$. Since

$$4x_1^2 - 9y_1^2 - 8x_1 + 36y_1 - 68 = 144 - 81 - 48 + 108 - 68 = 55 > 0$$

the point (6, 3) lies outside the hyperbola.

Example 17. *Reduce the following equations to the standard form of the equation of the hyperbola*

(i) $5x^2 - 4y^2 - 10x - 16y - 31 = 0$; (ii) $7y^2 - 9x^2 + 54x - 28y - 116 = 0$.

Hence, obtain the

(a) *equation of the axis;*

(b) *coordinates of the centre and the vertices;*

(c) *eccentricity and coordinates of the foci;*

(d) *lengths of the transverse axis, conjugate axis and the latus rectum;*

(e) *equations of the directrices and latus rectum.*

Solution: (i) We write the given equation in the form:

$$5(x^2 - 2x) - 4(y^2 + 4y) = 31$$

or $\quad 5(x^2 - 2x + 1) - 4(y^2 + 4y + 4) = 31 + 5 - 16$

or $\quad 5(x - 1)^2 - 4(y + 2)^2 = 20$

or $\quad \frac{(x-1)^2}{4} - \frac{(y+2)^2}{5} = 1$, or $\frac{X^2}{4} - \frac{Y^2}{5} = 1$

where $X = x - 1$ and $Y = y + 2$.

The equation represents a horizontal hyperbola.

Comparing it with the equation $\frac{X^2}{a^2} - \frac{Y^2}{b^2} = 1$

we get $a^2 = 4$ and $b^2 = 5$.

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4. $\lim_{x \to a} [cf(x)] = c \cdot \lim_{x \to a} f(x)$
   That is, the limit of a constant times a function is the constant times the limit.
5. If $h(x) = c$ for all x, then $\lim_{x \to a} h(x) = c$.
   That is, the limit of a constant function is a constant. This is usually addreviated by writing $\lim_{x \to a} c = c$.
6. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$, for any positive integer n.
7. $\lim_{x \to a} x^n = a^n$ for any positive integer n.

Remark: In Property 6, we require that $\lim_{x \to a} f(x)$ be positive, if n is even.

EVALUATION OF ALGEBRAIC LIMITS

The following methods are useful for evaluating algebraic limits:

Method of Factorization

If $f(x)$ and $g(x)$ are polynomials and $g(a) \neq 0$, then we know that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{f(a)}{g(a)}$$

Now, if $f(a) = 0 = g(a)$, then $(x - a)$ is a factor of both $f(x)$ and $g(x)$. We cancel this common factor $(x - a)$ from both the numerator and denominator and again put $x = a$ in the given expression. If we get a meaningful number then that number is the limit of the given expression, otherwise we repeat this process till we get a meaningful number.

Some useful results

(i) $x^2 - y^2 = (x - y)(x + y)$
(ii) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
(iii) $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$
(iv) $x^4 - y^4 = (x^2 - y^2)(x^2 + y^2) = (x + y)(x - y)(x^2 + y^2)$

Example 1. *Evaluate the following limits:*

(i) $\lim_{x \to 1/2} \frac{4x^2 - 1}{2x - 1}$; (ii) $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$;

(iii) $\lim_{x \to 2} \frac{x^3 - 8}{x^2 - 4}$; (iv) $\lim_{x \to 1} \frac{x - 1}{2x^2 - 7x + 5}$.

Solution: (i) $\lim_{x \to 1/2} \frac{4x^2 - 1}{2x - 1} = \lim_{x \to 1/2} \frac{(2x + 1)(2x - 1)}{2x - 1}$

[Cancelling the non-zero factor $(2x - 1)$]

$$= \lim_{x \to 1/2} (2x + 1) = 2\left(\frac{1}{2}\right) + 1 = 2$$

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21. $\lim_{x \to \pi} \dfrac{1+\cos x}{\tan^2 x}$

22. $\lim_{x \to \pi/4} \dfrac{\tan^3 x - \tan x}{\cos\left(x+\dfrac{\pi}{4}\right)}$

23. $\lim_{x \to \pi/2} \dfrac{\sqrt{2-\sin x}-1}{\left(\dfrac{\pi}{2}-x\right)^2}$

24. $\lim_{x \to \pi/3} \dfrac{\sqrt{3}-\tan x}{\pi - 3x}$

25. $\lim_{x \to \pi} \dfrac{\sin 3x - 3\sin x}{(\pi - x)^3}$

26. $\lim_{x \to \pi/2} \dfrac{\sec x - \tan x}{\left(\dfrac{\pi}{2}-x\right)}$

27. $\lim_{x \to \pi/2} \dfrac{\cos 3x + 3\cos x}{\left(\dfrac{\pi}{2}-x\right)^3}$

28. $\lim_{x \to \pi/6} \dfrac{2-\sqrt{3}\cos x - \sin x}{(6x-\pi)^2}$

29. $\lim_{x \to \pi/8} \dfrac{\cot 4x - \cos 4x}{(\pi - 8x)^3}$

30. $\lim_{x \to 2} \dfrac{x^2 - x - 2}{x^2 - 2x + \sin(x-2)}$

31. $\lim_{x \to a} \dfrac{\cos x - \cos a}{\cot x - \cot a}$

32. $\lim_{\theta \to \pi/4} \dfrac{\sin\theta - \cos\theta}{\tan\theta - \cot\theta}$

Answers

1. 1.	**2.** 1.	**3.** 1.	**4.** 0.	**5.** $\frac{1}{2}$.	**6.** 0.
7. $\frac{1}{4}$	**8.** –2.	**9.** $\frac{\sin\theta\cos\theta}{\theta}$.	**10.** $\cos y$.	**11.** $\sec^2 y$.	**12.** 2.
13. $-\frac{1}{2}$.	**14.** $-\sin 3$.	**15.** $-2\sqrt{a}\sin a$.	**16.** $\frac{1}{\sqrt{2}}$.	**17.** $-\frac{3}{2}$.	**18.** $\frac{1}{8}$.
19. $\frac{1}{\pi}$.	**20.** 2.	**21.** $\frac{1}{2}$.	**22.** –4.	**23.** $\frac{1}{4}$.	**24.** $\frac{4}{3}$.
25. –4.	**26.** $\frac{1}{2}$.	**27.** 4.	**28.** $\frac{1}{36}$.	**29.** $\frac{1}{16}$.	**30.** 1.
31. $\sin^3 a$.	**32.** $\frac{1}{2\sqrt{2}}$.				

HINTS AND SOLUTIONS

9. $$\lim_{x\to\theta}\frac{\sin^2 x - \sin^2\theta}{x^2-\theta^2} = \lim_{x\to\theta}\frac{\sin x + \sin\theta}{x+\theta}\lim_{x\to\theta}\frac{\sin x - \sin\theta}{x-\theta}$$

$$= \frac{\sin\theta+\sin\theta}{\theta+\theta}\lim_{x\to\theta}\frac{2\cos\frac{x+\theta}{2}\sin\frac{x-\theta}{2}}{x-\theta}$$

$$= \frac{2\sin\theta}{2\theta}(2)\lim_{x\to\theta}\cos\frac{x+\theta}{2}\lim_{x\to\theta}\frac{\sin\frac{x-\theta}{2}}{\frac{x-\theta}{2}}\left(\frac{1}{2}\right)$$

$$= \frac{2\sin\theta}{\theta}\cos\left(\frac{\theta+\theta}{2}\right)(1)\left(\frac{1}{2}\right) = \frac{\sin\theta\cos\theta}{\theta}$$

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Example 10. *Let a function f(x) be defined as* $f(x) = \begin{cases} x, & \text{if } 0 \le x \le 1/2 \\ 0, & \text{if } x = 1/2 \\ x-1, & \text{if } 1/2 < x \le 1 \end{cases}$

Establish the existence of $\lim_{x \to 1/2} f(x)$.

Solution: R.H.L. = $\lim_{x \to \frac{1}{2}+} f(x) = \lim_{x \to \frac{1}{2}+} (x-1)$

Put $x = \frac{1}{2} + h$ so that when $x \to \frac{1}{2}$, $h \to 0$.

$\therefore$ R.H.L. = $\lim_{h \to 0}\left(\frac{1}{2} + h - 1\right) = \frac{1}{2} + 0 - 1 = -\frac{1}{2}$.

Also, L.H.L. = $\lim_{x \to \frac{1}{2}-} f(x) = \lim_{x \to \frac{1}{2}-} x$

Put $x = \frac{1}{2} - h$, so that when $x \to \frac{1}{2}$, $h \to 0$.

$\therefore$ L.H.L. = $\lim_{h \to 0}\left(\frac{1}{2} - h\right) = \frac{1}{2} - 0 = \frac{1}{2}$.

Since R.H.L. $\ne$ L.H.L.

Therefore, $\lim_{x \to 1/2} f(x)$ does not exist.

Example 11. *If* $f(x) = \begin{cases} \cos x & x \ge 0 \\ x + k, & x < 0 \end{cases}$, *find the value of k, given that* $\lim_{x \to 0} f(x)$ *exists.*

Solution: L.H.L. = $\lim_{x \to 0-} f(x) = \lim_{x \to 0-} (x+k)$

Put $x = 0 - h$. As $x \to 0-$, $h \to 0$.

$\therefore$ L.H.L. = $\lim_{h \to 0} (0 - h + k) = (0 - 0 + k) = k$

R.H.L. = $\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \cos x$

Put $x = 0 + h$. As $x \to 0+$, $h \to 0$.

$\therefore$ R.H.L. = $\lim_{h \to 0} \cos(0 + h) = \cos 0 = 1$

Since $\lim_{x \to 0} f(x)$ exists,

$\therefore$ L.H.L. = R.H.L.

$\Rightarrow$ $k = 1$

EXERCISE 13.5

LEVEL OF DIFFICULTY A

1. Evaluate the following one sided limits:

(i) $\lim_{x \to 2^+} \frac{x-3}{x^2-4}$;

(ii) $\lim_{x \to 2^-} \frac{x-3}{x^2-4}$;

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(iv) $\dfrac{1}{px^2+qx+r}$; (v) $\dfrac{\sin x + x\cos x}{x\sin x - \cos x}$; (vi) $\dfrac{\sec x - 1}{\sec x + 1}$;

(vii) $\dfrac{\cos x}{\sin x + \cos x}$; (viii) $\dfrac{\sqrt{a}+\sqrt{x}}{\sqrt{a}-\sqrt{x}}$; (ix) $\dfrac{(x-1)(x-2)}{(x-3)(x-4)}$.

2. (i) If $f(x) = \dfrac{1-\cos x}{1+\cos x}$, find $f'\left(\dfrac{\pi}{2}\right)$.

(ii) If $f(x) = \dfrac{x\sin x}{\cos x - \sin x}$, find $f'(0)$.

3. If $y = \dfrac{x}{x+a}$, prove that $x\dfrac{dy}{dx} = y(1-y)$.

4. If $f(x) = \dfrac{\operatorname{cosec} x - \cot x}{\operatorname{cosec} x + \cot x}$, find $f'(x)$.

5. Differentiate $\dfrac{x+2}{x^2-3}$ and find the value of the derivative at $x = 0$.

Answers

1. (i) $-\dfrac{65}{(4x-7)^2}$;

(ii) $\dfrac{(n-1)x^n - a(nx^{n-1} - a^{n-1})}{(x-a)^2}$; (iii) $\dfrac{-12(x^2-2x+6)}{(x^2-9x+3)^2}$;

(iv) $-\dfrac{2px+q}{(px^2+qx+r)^2}$; (v) $-\dfrac{2+x^2}{(x\sin x - \cos x)^2}$;

(vi) $\dfrac{2\sec x\tan x}{(\sec x+1)^2}$; (vii) $\dfrac{1}{1+\sin 2x}$;

(viii) $\dfrac{\sqrt{a}}{\sqrt{x}\left(\sqrt{a}-\sqrt{x}\right)^2}$; (ix) $\dfrac{-4x^2+20x-22}{(x-3)^2(x-4)^2}$.

2. (i) 2; (ii) 0; (iii) $\dfrac{1}{2}\left(2\sqrt{2}e^{\pi/4} - \sqrt{2}e^{\pi/4} - 1\right)$.

4. $\dfrac{2\operatorname{cosec} x}{(\operatorname{cosec} x + \cot x)^2}$. **5.** $-\dfrac{x^2+4x+3}{(x^2-3)^2}$, $-\dfrac{1}{3}$.

Example 10. *Calculate the mean deviation from the median of the following data:*

Wages per week (in Rs)	10–20	20–30	30–40	40–50	50–60	60–70	70–80
No. of workers	4	6	10	20	10	6	4

Solution: We construct the following table:

Wages per week in (Rs)	Mid-value (x_i)	Frequency f_i	Cumulative frequency	$\lvert d_i \rvert = \lvert x_i - 45 \rvert$	$f_i \lvert d_i \rvert$
10–20	15	4	4	30	120
20–30	25	6	10	20	120
30–40	35	10	20	10	100
40–50	45	20	40	0	0
50–60	55	10	50	10	100
60–70	65	6	56	20	120
70–80	75	4	60	30	120
		$N = \Sigma f_i = 60$			$\Sigma f_i \lvert d_i \rvert = 680$

Here $N = 60$. So, $\frac{N}{2} = 30$.

The cumulative frequency just greater than $\frac{N}{2} = 30$ is 40 and the corresponding class is 40–50. So, 40–50 is the median class.

$\therefore \qquad l = 40, \quad f_c = 20, \quad h = 10, \quad f_m = 20$

Now,

$$\text{Median} = l + \frac{\frac{N}{2} - f_c}{f_m} \times h$$

$$= 40 + \frac{30 - 20}{20} \times 10 = 45$$

$$\text{Mean deviation from median} = \frac{\Sigma f_i |d_i|}{N} = \frac{680}{60} = 11.33$$

Example 11. *Calculate the mean deviation from the median for the following data:*

Wages per day	*No. of workers*
20–0	3
30–40	8
40–50	12
50–60	9
60–70	8

For shares X:

$$\text{Mean } \bar{x} = A + \frac{\Sigma y_i}{10} = 52 + \frac{-10}{10} = 51$$

$$\text{S.D.} = \sigma = \sqrt{\frac{1}{N^2}[N\Sigma y_i^2 - (\Sigma y_i)^2]}$$

$$= \frac{1}{N}\sqrt{10 \times 360 - (-10)^2}$$

$$= \frac{1}{10}\sqrt{3600 - 100}$$

$$= \frac{\sqrt{3500}}{10} = \frac{59.16}{10} = 5.916$$

$$\text{Coefficient of variation} = \frac{\sigma}{\bar{x}} \times 100 = \frac{5.916}{51} \times 100$$

$$= 11.6$$

For shares Y:

$$\text{Mean } \bar{x} = A + \frac{\Sigma y_i}{10} = 105 + 0 = 105$$

$$\text{S.D.} = \sigma = \frac{1}{N}\sqrt{N\Sigma y_i^2 - (\Sigma y_i)^2}$$

$$= \frac{1}{10}\sqrt{10 \times 40 - 0}$$

$$= \frac{20}{10} = 2$$

$$\text{Coefficients of variation} = \frac{\sigma}{\bar{x}} \times 100$$

$$= \frac{2}{105} \times 100$$

$$= 1.9$$

Coefficient of variation in shares Y is less than the coefficient of variation in shares X.

Therefore, the share Y is more stable than the share X.

Example 3. *An analysis of monthly wages paid to workers in two firms A and B belonging to the same industry, give the following results:*

	Firm A	***Firm B***
No. of wages earned	586	648
Mean of monthly wages	Rs 5253	Rs 5253
Variance of distribution of wages	100	121

(i) *Which firm A or B pays out larger amount as monthly wages?*

(ii) *Which firm A or B, shows greater variability in individual wages?*

$A \cup B$ is the event of getting an odd number or a number greater than 2.

$$A \cap B = \{3, 5\}$$

$A \cap B$ is the event of getting an odd number greater than 2.

$$\overline{A} = \{2, 4, 6\} \quad \text{[Those elements of } S \text{ which are not in } A.]$$

$\overline{A}$ is the event of not getting an odd number, i.e., getting an even number.

$$\overline{B} = \{1, 2\}$$

$\overline{B}$ is the event of not getting a number greater that 2, i.e., getting a number less than or equal to 2.

$$\overline{A} \cap \overline{B} = \{2\}$$

$\overline{A} \cap \overline{B}$ is the event of neither getting an odd number nor a number greater than 2.

Mutually Exclusive Events

In an experiment, if the occurrence of an event precludes or rules out the happening of all the other events in the same experiment.

Illustration 10.

(i) When a coin is tossed either head or tail will appear. Head and tail cannot appear simultaneously. Therefore, occurrence of a head or a tail are two mutually exclusive events.

(ii) In throwing a die all the 6 faces numbered 1 to 6 are mutually exclusive since if any one of these faces comes, the possibility of others in the same trial, is ruled out.

Note: A and B are mutually exclusive events $\Leftrightarrow A \cap B = \phi$, i.e., A and B are disjoint sets.

Illustration 11.

(i) If the random experiment is 'a die is thrown' and A, B are the events, A : the number is less than 3; B : the number is more than 4, then $A = \{1, 2\}$, $B = \{5, 6\}$.

$A \cap B = \phi$, thus A and B are mutually exclusive events.

(ii) If the random experiment is 'a card is drawn from a well-shuffled pack of cards' and A, B are the events A : the card is Black; B: the card is an ace.

Since a black card can be an ace, $A \cap B \neq \phi$, thus A and B are not mutually exclusive events.

Mutually Exclusive and Exhaustive Events

Events $E_1, E_2, ..., E_n$ are called *mutually exclusive and exhaustive* if $E_1 \cup E_2 \cup ... \cup E_n = S$, i.e., $\bigcup_{i=1}^{n} E_i = S$ and $E_i \cap E_j = \phi$ for all $i \neq j$.

For example, in a single throw of a die, let A be the event of getting an even number and B be event of getting odd numbers, then

$$A = \{2, 4, 6\}, B = \{1, 3, 5\}$$

$$A \cap B = \phi, A \cup B = \{1, 2, 3, 4, 5, 6\} = S$$

$\therefore$ A and B mutually exclusive and exhaustive events.

Example 1. (i) *If one coin is tossed, what is the resulting sample space?*

(ii) *If two coins are tossed simultaneously, what is the resulting sample space?*

(iii) *If three coins are tossed simultaneously, what is the resulting sample space?*

Example 13. *In a single throw of three dice, determine the probability of getting*
(i) *a total of 5* (ii) *a total of at most 5.* (iii) *a total of at least 5.*

Solution: Number of exhaustive cases in a single throw of three dice = $6 \times 6 \times 6 = 216$

(i) Cases favourable to a total of 5 are (1, 2, 2), (2, 1, 2), (2, 2, 1), (1, 1, 3), (1, 3, 1), (3, 1, 1).

Their number = 6

$$\therefore \qquad P(\text{a total of } 5) = \frac{6}{216} = \frac{1}{36}$$

(ii) *A* total of at most 5 means a total of 3, 4 or 5.

Cases favourable to a total of 3 are (1, 1, 1).

Cases favourable to a total of 4 are (1, 1, 2), (1, 2, 1), (2, 1, 1).

Cases favourable to a total of 5 are (1, 2, 2), (2, 1, 2), (2, 2, 1), (1, 1, 3), (1, 3, 1), (3, 1, 1).

Number of cases favourable to a total of 3 or 4 or 5 is 10.

$$\therefore \qquad P(\text{a total of at most } 5) = \frac{10}{216} = \frac{5}{108}$$

(iii) *A* total of at least 5 means not a total of 3 or 4.

Number of cases favourable to a total of 3 or 4 is 4.

$$P(\text{a total of 3 or 4}) = \frac{4}{216} = \frac{1}{54}$$

$$\begin{aligned} \therefore \qquad P(\text{a total of at least } 5) &= P(\text{not a total of 3 or 4}) \\ &= 1 - P(\text{a total of 3 or 4}) \\ &= 1 - \frac{1}{54} = \frac{53}{54} \end{aligned}$$

Example 14. *What is the chance that a leap year, selected at random will contain 53 Sundays?*

Solution: We know that a leap year has 366 days and thus a leap year has 52 weeks and 2 days over.

The two over (successive) days have the following likely cases:

(i) Sunday and Monday; (ii) Monday and Tuesday;
(iii) Tuesday and Wednesday; (iv) Wednesday and Thursday;
(v) Thursday and Friday; (vi) Friday and Saturday;
(vii) Saturday and Sunday.

$\therefore$ Number of exhaustive cases $n = 7$.

Out of these the favourable cases are (i) and (vii).

$\therefore$ Number of favourable cases $m = 2$

$\therefore$ Probability of having 53 Sundays $= \dfrac{2}{7}$

Example 15. *What is the probability that a number selected from the numbers* 1, 2, ..., 25 *is a prime number? You may assume that each of the* 25 *numbers is equally likely to be selected.*

Theorem 5. *For any two events E and F,* $P(E-F) = P(E) - P(E \cap F)$.

Proof. Let E and F be two compatiable events.

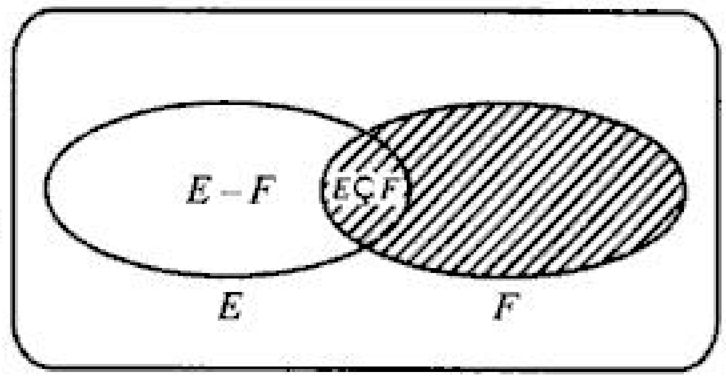

Fig. 16.1

From the Venn diagram shown in Fig. 16.1, it is clear that $(E-F) \cap (E \cap F) = \phi$ and $(E-F) \cup (E \cap F) = E$

$\therefore \quad P(E) = P[(E-F) \cup (E \cap F)]$

$= P(E-F) + P(E \cap F)]$ $[\because (E-F) \cap (E \cap F) = \phi]$

Hence $\quad P(E-F) = P(E) - P(E \cap F)$

Theorem 6. (Addition theorem). *For any two events E and F,*

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

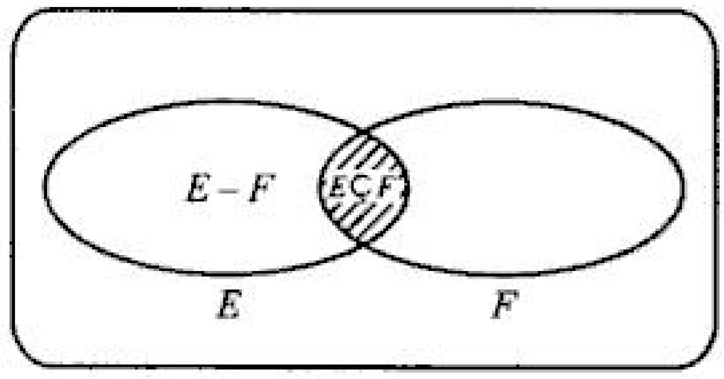

Fig. 16.2

Proof. Let S be the sample space and let E and F be any two events. Then

$$E \subseteq S \text{ and } F \subseteq S$$

From the Venn diagram given in Fig. 16.2, it is clear that $(E-F) \cap F = \phi$

and $\quad (E-F) \cup F = (E \cup F)$

$\therefore \quad P(E \cup F) = P[(E-F) \cup F)]$

$= P(E-F) + P(F) \quad [\because (E-F) \cap F = \phi]$

$= P(E) - P(E \cap F) + P(F) \quad [\because P(E-F) = P(E) - P(E \cap F)]$

Hence $\quad P(E \cup F) = P(E) + P(F) - P(E \cap F)$

Notes: 1. We may express the above results as $P(E \text{ or } F) = P(E) + (F) - P(E \text{ and } F)$

2. If E and F are mutually exclusive, then

$$P(E \cap F) = 0 \text{ and so } P(E \cup F) = P(E) + P(F)$$

Theorem 7. *If E_1 and E_2 be two events such that $E_1 \subseteq E_2$, then prove that $P(E_1) \leq P(E_2)$.*

Proof. Since $E_1 \subseteq E_2$, it is clear from the Venn diagram shown in Fig. 16.3 that

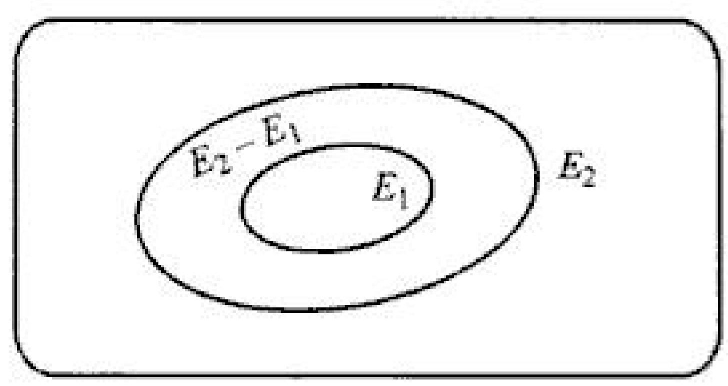

Fig. 16.3

$$E_2 = E_1 \cup (E_2 - E_1)$$

and $\quad E_1 \cap (E_2 - E_1) = \phi$

$\therefore \quad P(E_2) = P[(E_1 \cup (E_2 - E_1)]$

$= P(E_1) + P(E_2 - E_1)$

$[\because E_1 \cap (E_2 - E_1) = \phi]$

$\therefore \quad P(E_1) \leq P(E_2)$

$[\because P(E_2 - E_1) \geq 0]$

Theorem 8. If E is an event associated with a random experiment, then $0 \leq P(E) \leq 1$.

Proof. Let S be the sample space and let E be an event.

Then, $\phi \subseteq E$ and $E \subseteq S \Rightarrow P(\phi) \leq P(E)$ and $P(E) \leq P(S)$

$\Rightarrow \quad 0 \leq P(E)$ and $P(E) \leq 1 \quad [\because P(\phi) = 0 \text{ and } P(S) = 1]$

Hence $\quad 0 \leq P(E) \leq 1.$

Now, $P(\overline{A}) + P(\overline{B}) = 1 - P(A) + 1 - P(B)$
$= 2 - [P(A) + P(B)] = 2 - 0.9 = 1.1.$

Example 24. *The probabilities of happening of two events A and B are* 0.25 *and* 0.50 *respectively. If the probability of happening of A and B together is* 0.14, *find the probability that neither A nor B happens.*

Solution: Required probability $= 1 - P(A \cup B)$
$= 1 - [P(A) + P(B) - P(A \cap B)]$
$= 1 - [0.25 + 0.50 - 0.14]$
$= 1 - [0.61] = 0.39$

Example 25. *The probability of an event A occurring is* 0.5 *and that of B is* 0.3. *If A and B are mutually exclusive events, find the probability that neither A nor B occurs.*

Solution: Required probability $= 1 - P(A \cup B)$
$= 1 - [P(A) + P(B) - P(A \cap B)]$
$= 1 - [0.5 + 0.3 - 0]$
[$\because$ A and B are mutually exclusive events]
$= 1 - 0.8 = 0.2$

Example 26. *If A, B, C are three events such that*

$$P(A) = 0.3,\ P(B) = 0.4,\ P(C) = 0.8,\ P(A \cap B) = 0.08$$
$$P(A \cap C) = 0.28,\ P(A \cap B \cap C) = 0.09.$$

If $P(A \cup B \cup C) \geq 0.75$, *then show that* $0.23 \leq P(B \cap C) \leq 0.48$.

Solution: We know that

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

$\therefore$ $$P(A \cup B \cup C) = 0.3 + 0.4 + 0.8 - 0.08 - P(B \cap C) - 0.28 + 0.09$$
$$= 1.23 - P(B \cap C)$$

Since $P(A \cup B \cup C) \geq 0.75$ (given)

$\therefore$ $1.23 - P(B \cap C) \geq 0.75 \Rightarrow P(B \cap C) \leq 0.48$ ...(1)

Also $P(A \cup B \cup C) \leq 1$

$\therefore$ $1.23 - P(B \cap C) \leq 1 \Rightarrow P(B \cap C) \geq 0.23$ ...(2)

Combining (1) and (2), we get

$$0.23 \leq P(B \cap C) \leq 0.48.$$

Example 27. *The probabilities that a student will receive an A, B, C or D grade are* 0.30, 0.38, 0.22 *and* 0.10 *respectively. What is the probability that student will receive*

(i) *at least B grade;* (ii) *atmost C grade;*
(iii) *not an A grade;* (iv) *B or C grade.*

Solution: (i) P(at least B grade) $= P(B$ grade$) + P(A$ grade$)$
$= 0.38 + 0.30 = 0.68$

(ii) P(at most C grade) $= P(C$ grade$) + P(D$ grade$)$
$= 0.22 + 0.10 = 0.32$

41. Probability is 0.45 that a dealer will sell atleast 20 television sets during a day and the probability is 0.74 that he will sell less than 24 televisions.

What is the probability that he will sell 20, 21, 22, or 23 televisions during the day?

42. The probabilities of a student getting first class, second class or third class at an examination are 1/10, 3/5, 1/4 respectively. Find the probability that he fails.

Answers

1. (i) $\frac{5}{8}$; (ii) $\frac{3}{8}$. **2.** 0.85.

3. (i) 0.58; (iii) 0.52; (iii) 0.74.

4. (i) $\frac{7}{13}$; (ii) $\frac{1}{2}$; (iii) $\frac{2}{13}$; (iv) $\frac{2}{13}$; (v) $\frac{1}{2}$; (vi) $\frac{9}{13}$.

5. $\frac{5}{8}$. **6.** (i) $\frac{5}{18}$; (ii) $\frac{13}{18}$.

7. $\frac{2}{3}$. **8.** (i) 0.6; (ii) 0.5; (iii) 0.25.

9. $\frac{1}{4}$. **10.** 0.08. **11.** $\frac{4}{5}$. **12.** $\frac{31}{36}$.

13. $\frac{7}{8}$. **14.** (i) $\frac{1}{2}$; (ii) $\frac{1}{2}$; (iii) $\frac{5}{6}$.

15. $\frac{13}{30}$. **16.** $\frac{17}{105}$. **17.** $\frac{7}{17}$.

18. (i) $\frac{19}{30}$ $[P(A \cup B) = P(A) + P(B) - P(A \cap B)]$; (ii) $0.37[P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)]$;

(iii) $\frac{5}{15}$ $[P(\bar{A} \cap \bar{B}) = P(B) - P(A \cap B)]$.

19. $\frac{13}{36}$. **20.** $\frac{6}{11}$. **21.** 1.2. **22.** 1.2

23. $\frac{13}{18}$. **24.** 0.39. **25.** (i) $\frac{4}{9}$; (ii) $\frac{5}{9}$.

26. 0.65 **27.** (i) $\frac{1}{2}$; (ii) $\frac{1}{6}$; (iii) $\frac{1}{12}$; (iv) $\frac{1}{3}$; (v) $\frac{1}{4}$.

28. $\frac{5}{9}$. **29.** $\frac{11}{36}$. **30.** $\frac{4}{9}$. **31.** $\frac{4}{9}$.

32. 0.30. **33.** $\frac{2}{20}, \frac{1}{10}$. **34.** $\frac{25}{52}$. **35.** 3 : 2.

36. 0.2. **37.** $\frac{-1}{4} \le p \le \frac{1}{3}$. **38.** $\frac{4}{7}$. **39.** $\frac{5}{12}$.

40. 0.8. **41.** 0.19. **42.** $\frac{1}{20}$.

HINTS AND SOLUTIONS

1. (ii) $P(\overline{A} \cap \overline{B}) = 1 - P(A \cup B)$

7. Sample space S = {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}

Number of exhaustive cases = 12 i.e., $n(S) = 12$

Let A be the event 'a head'. Then

$$A = (\text{H1, H2, H3, H4, H5, H6,}) \Rightarrow n(A) = 6$$

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{6}{12}$$

Let B be the event 'a number greater than 4'. Then,

$$B = \{\text{H5, H6, H4, H6}\} \Rightarrow n(B) = 4$$

$$\therefore \quad P(B) = \frac{n(B)}{n(S)} = \frac{4}{12}$$

Favourable cases to $A \cap B$ = {H5, H6}

$$\therefore \quad P(A \cap B) = \frac{2}{12}$$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{6}{12} + \frac{4}{12} - \frac{2}{12} = \frac{8}{12} = \frac{2}{3}$

i.e. Probability (a head or no. > 4 or both) $= \frac{2}{3}$

9. In the first two hundred digits 6, 12, 18, ..., 198 are divisible by 6.

Their number $= \frac{198}{6} = 33$.

8, 16, 24, ..., 200 are divisible by 8

Their number $= \frac{200}{8} = 25$.

24, 48, 72, ..., 192 are divisible by 24

Their number $= \frac{192}{24} = 8$.

$$\therefore \quad P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

$$= \frac{33}{200} + \frac{25}{200} - \frac{8}{200} = \frac{8}{200} = \frac{50}{200} = \frac{1}{4}$$

12. P(passes in atleast one subject)

$$= P(M \cup E) = P(M) + P(E) - P(M \cap E) = \frac{2}{3} + \frac{4}{9} - \frac{1}{4} = \frac{31}{36}$$

13. When three coins are tossed, the sample space S is

$$S = (\text{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT})$$

$$\Rightarrow \quad n(S) = 8$$

Let A and B denote the events of at the most two tails and at least two heads.

$\therefore$ A : HHT, HTH, THH, HTT, THT, TTH i.e., $n(A) = 6$

B : HHT, HTH, THH, HHH i.e., $n(B) = 4$

and $A \cap B$: HHT, HTH, THH i.e., $n(A \cap B) = 3$

$$\therefore \quad P(A) = \frac{n(A)}{n(S)} = \frac{6}{8},\ P(B) = \frac{n(B)}{n(S)} = \frac{4}{8} \text{ and } P(A \cap B) = \frac{3}{8}$$

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The Authors

DINESH KHATTAR (Ph.D.), an alumnus of Hindu College, is with the Department of Mathematics at Kirori Mal College, University of Delhi. Besides teaching UG classes at Kirori Mal College for more than 20 years, he is also taking PG classes at the University. He is a topper (Gold Medalist) in his B.Sc. and M.Sc. exams. Dr. Khattar is an author of many books on mathematics including *CBSE Mathematics for Class XII* (Part 1 and Part 2).

ANITA KHATTAR (M.Sc., B.Ed.), an alumna of Hindu College, University of Delhi, is a mathematics teacher at Sarvodaya Vidyalaya, I.P. Extension, Delhi for more than 17 years. She has co-authored a number of books on mathematics including *CBSE Mathematics for Class XII* (Part 1 and Part 2).

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